Linear groups acting 4-arc-transitively on cubic graphs.

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Jt. w/ Michael Giudici

Goal: explain title, then ignore again

Defn

Let Γ be a graph with vertex set $V(\Gamma)$. An s-arc in Γ is an $(s+1)$-tuple $v_0, \dots, v_s$ for $v_i \in V(\Gamma)$ s.t. $v_i \sim v_{i+1}$, $v_i \neq v_{i+2}$ $\forall i$.

Defn Let $G \leq \mathrm{Aut}\,\Gamma$. We say Γ is (G, s)-arc-transitive (a-t.) if G acts transitively on the s-arcs in Γ.

Say Γ is s-a-t. if such a G exists.

Propn Suppose Γ is s-a-t. and every $(s-1)$-arc in Γ extends to an s-arc. Then Γ is $(s-1)$-a-t.

Note: - If Γ is a connected s-a-t. then Γ is regular

- Cycles are s-a-t. $\forall s$.

Thm (Tutte '47, '59) Γ cubic (3-regular) s-a-t. $\Rightarrow s \leq 5$.

Thm (Weiss '81) Γ s-a-t. $\Rightarrow s \leq 7$.

Praeger '93: Study s-a-t. graphs for $s \geq 2$

$\rightsquigarrow G \leq \mathrm{Aut}\,\Gamma$ quasiprimitive or bi-qp. $\rightarrow$ $N \trianglelefteq G \Rightarrow N$ transitive.

$\rightsquigarrow$ O'Nan-Scott type thm of quasiprimitive gps $\Rightarrow$ 8 types, 4 arise.

Ivanov-Praeger '93   Baddeley '93

$\underbrace{\text{HA}, \quad \text{TW},}_{\substack{s \leq 3. \\ \text{Classified.}}}$ AS, PA $\longrightarrow$ reduces to simple.

Classified.

So to investigate S-a-t. graphs for $S \geq 4$, look at almost simple gps.

<u>Th$\underline{m}$</u> (Djoković – Miller, '80)
Let Γ be a cubic graph and $v, w \in V(\Gamma)$. If Γ is (G,S)-a-t. then the structure of G_v, $G_{\{v,w\}}$ and $G_{v,w} = G_v \cap G_{\{v,w\}}$ is known.

If $S = 4$, $\quad G_v \cong S_4$, $\quad G_{\{v,w\}} \cong D_{16}$ or SD_{16} and $G_{v,w} \cong D_8$.

<u>Lemma</u>
Γ is $(G,4)$-a-t. cubic graph $\iff$ $\exists$ subgps G_v, $G_{\{v,w\}}$, $G_{v,w}$ as above s.t. $G = \langle G_v, G_{\{v,w\}} \rangle$.

($\Leftarrow$ from coset graphs)

Q: Which almost simple groups $T \leq G \leq \mathrm{Aut}\, T$ are such that

$$G = \langle A, B \rangle \text{ for } A \cong S_4,\ B \cong D_{16},\ A \cap B \cong D_8.$$

Goal: Above holds for $PSL_n(q)$ for $(q,6) = 1$ and $n \geq 400$.

Generation: $G = \langle A, B \rangle \iff \nexists M \leq_{max} G$ s.t. $A, B \leq M$.

$\rightsquigarrow$ Max subgps matter!

Max subgps known by Aschbacher's Th$\underline{m}$. geometric

– 9 classes $\mathcal{C}_1 - \mathcal{C}_8$ + $S = \mathcal{C}_9$.

<u>What do we do?</u>

Embed A, B in $SL_n(q)$ with a rep chosen to avoid some $\mathcal{C}_i$'s, and to ensure $A \cap B = C \cong D_8$. Let $G := GL_n(q) = GL(V)$

$S_4 \quad D_{16} \quad D_8$.

$S_4 \quad D_{16} \quad D_8$.

Fix A, B, C as above.

Collection of D_{16}s containing C.

$$\mathcal{B} := \{ B^g \mid g \in N_G(C) \}$$

$$\mathcal{B}_i := \{ B^g \in \mathcal{B} \mid \exists H \in \mathcal{C}_i \text{ s.t. } \langle A, B^g \rangle \leq H \} = \text{"} \mathcal{B} \cap \mathcal{C}_i \text{"}$$

$$\mathcal{B}_H := \{ B^g \in \mathcal{B} \mid \langle A, B^g \rangle \leq H^h \text{ for } h \in G \} = \text{"} \mathcal{B} \cap H^G \text{"}$$

We aim to show

$$\mathcal{B} \neq \bigcup_{i=1}^{9} \mathcal{B}_i .$$

In fact,
$$\frac{\left| \mathcal{B} \setminus \bigcup_{i=1}^{9} \mathcal{B}_i \right|}{|\mathcal{B}|} \to 1 \quad \text{as } n \to \infty \quad \text{or } q \to \infty .$$

If so, $\exists B^g \in \mathcal{B}$ s.t. $\not\exists M \leq_{\max} G$ s.t. $\langle A, B^g \rangle \leq M$, so $\langle A, B^g \rangle = SL_n(q)$.

Lemma

$$|\mathcal{B}| \geq \frac{|N_G(C)|}{|N_G(B)|} \sim 2^{-1/4} q^{n^2/16} .$$

Otherwise, we use the following:

Lemma A $\langle A, B \rangle \leq X \trianglelefteq Y \Rightarrow A \cap X = B \cap X = 1$ or $A \leq X$ and $B \cap X \geq C$.

Proof Suppose $A \cap X = 1$. Then $C \cap X = 1$. But $B \cap X \trianglelefteq B$, so $B \cap X = 1$.

Now let $A \cap X \neq 1$. Since $A \cap X \trianglelefteq A$, $V_4 \leq A \cap X$. Then $V_4 \leq B \cap X$.

Since $B \cap X \trianglelefteq B$, $C \leq B \cap X$. Then $C \leq A \cap X$.

Since $A \cap X \trianglelefteq A$, $A = A \cap X$. $\square$

B Let $H \leq G$ and $A_H := \{ A^g \in A^G \mid A^g \leq H \} = \text{"} H \cap A^G \text{"}$

Then $|\mathcal{B}_H| \leq |N_G(A)| \cdot |A_H|$

C/ Let H be a group. Then the n^{o} of subgps of H iso. to S_4 is at most $|H|^2$.

The Classes.

$\mathcal{C}_1$ $H \in \mathcal{C}_1$, Stab. $U \leq V$. $H \sim Q \rtimes (GL(U) \times GL(V/U))$

$N_H(C)$ acts on $\{\mathcal{B}^g \in \mathcal{B} \mid B^g \leq H\}$: We must determine how many orbits there are and how long they are.

orbit - stabiliser.

How do G-classes split in H?

Lemma Let X be a group and $\varphi, \psi : X \to GL(V)$ be quasi-equivalent, completely reducible reps of X stabilising $U \leq V$. Then $\varphi(X)$ and $\psi(X)$ are conj. in H iff $\varphi|_U$ and $\psi|_U$ are quasi-equivalent.

$$\rightsquigarrow \leq \frac{3n^3}{128} \text{ orbits} \rightsquigarrow |\mathcal{B}_1| \leq \frac{3n^3}{128} \sum_{\substack{\text{iso types} \\ A\text{-submodules } U \text{ of } V}} \frac{|N_H(C)|}{|N_H(B)|} \rightsquigarrow |\mathcal{B}_1| \leq 2^{70-\frac{1}{8}} |\mathcal{B}|.$$

$\mathcal{C}_3$ $H \in \mathcal{C}_3$ Stab. $F \geq \mathbb{F}_q$ of prime deg. r. $H \sim GL_{n/r}(q^r) \rtimes r =: X \rtimes r$.

Lemma $\mathcal{B}_3 = \emptyset$.

Proof Lemma $A \Rightarrow A \leq X$, $B \cap X \geq C$. $\square$

$\mathcal{C}_7$ $H \in \mathcal{C}_7$, Stab. $V = V_1 \otimes \dots \otimes V_t$ $\dim V_i = \sqrt[t]{n}$ $H \sim GL(V_1) \wr S_t$.

$\mathcal{C}_7$: $H \in \mathcal{C}_7$, Stab. $V = V_1 \otimes \dots \otimes V_t$ $\dim V_i = q^n$ $H \sim GL(V_1) \wr S_t$.

Lemma $|\mathcal{B}_7| \leq 2^{-\frac{n}{8}} |\mathcal{B}|$.

Proof: Lemma A $\Rightarrow A \cap X = B \cap X = 1 \Rightarrow t \geq 8$. $\Rightarrow H$ small.

Lemma B+C $\Rightarrow |\mathcal{B}_H| \leq |N_G(A)| \, |H|^2$. $\square$.

$\mathcal{S}$: $H \in \mathcal{S} \Rightarrow H$ almost simple

Liebeck: $|H| \leq q^{2n+4}$ Cameron-Neumann-Teague: Count n^o of sgs of order $\leq x$.

$\rightsquigarrow |\mathcal{B}_9| \leq 2^{-\frac{n}{8}} |\mathcal{B}|$.

$\mathcal{C}_2$: $H \in \mathcal{C}_2$ Stab. $V = V_1 \oplus \dots \oplus V_t$, $\dim V_i = \frac{n}{t}$. $H \sim GL_{\frac{n}{t}}(q) \wr S_t$.

$= X \rtimes t$.

Lemma A $\Rightarrow A \cap X = B \cap X = 1$. $\Rightarrow t \geq 8$.

In general, determining how A^G splits in H is hard.

Let $\pi: H \to S_t$. The n^o of subgps $Y \leq H$ s.t. $Y \cong S_4$ and $\pi(Y) = \pi(A)$. is at most $|\mathrm{Der}(A, X)|$.

$\rightarrow \delta: A \to X$ s.t. $\delta(gh) = \delta(g)\delta(h)^g$.

Looking at these we can show

$$|\mathcal{B}_H| \leq (t!)^2 \, |GL_{\frac{n}{t}}(q)|^{2t_1 + \frac{3}{2}t_2 + \sum_{i=3}^{24} t_i}$$

Where A has l_i orbits of length i as an element of S_t. i.e. acting on

$V = V_1 \oplus \dots \oplus V_t$.