# Brauer characters of $p$-solvable groups

郭青宏(Central South University)

This is a joint work with Weijun Liu (刘伟俊)

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- We always assume that $G$ is a finite group, and fix a prime number $p$.
- Let $K$ be a field. A **$K$-representation** of $G$ is just an $KG$-module $V$, or equivalently, a group homomorphism

$$\rho : G \to \mathrm{GL}_K(V) \cong \mathrm{GL}(n, K), \text{where } n = \dim_K(V).$$

- $\mathrm{char}K \nmid |G|$: **ordinary representation**. If $K$ is a splitting field for $G$, the number of simple $KG$-modules equals the number of conjugacy classes of $G$.

  The $KG$-modules are completely determined by their characters, i.e. if $V, W$ are two $KG$-modules, then $V \cong W$ if and only if $\chi_V = \chi_W$.

- char$K = p \mid |G|$: **modular representation**. If $K$ is a splitting field for $G$, the number of simple $KG$-modules equals the number of $p$-regular conjugacy classes of $G$.
- R. Brauer introduced the notion of **modular characters**, now known as **Brauer characters**, to understand the interplay between the representation theory in characteristic $p$ and ordinary character theory.

- Let $\mathbf{R}$ be the ring of algebraic integers in $\mathbb{C}$. We choose a maximal ideal $M$ of $\mathbf{R}$ containing $p\mathbf{R}$.
- Let $F = \mathbf{R}/M$ and let $^* : \mathbf{R} \to F$ be the canonical ring homomorphism.
- $S = \{rs^{-1} \mid r \in \mathbf{R}, s \in \mathbf{R} - M\},\ \mathcal{P} = \{rs^{-1} \mid r \in M, s \in \mathbf{R} - M\} = \mathbf{J}(S)$
- Notice that the map $^* : \mathbf{R} \to F$ can be extended to $S$ in a natural way. If $r \in \mathbf{R}$ and $s \in \mathbf{R} - M$, then $(rs^{-1})^* = r^*(s^*)^{-1}$.
- $\mathbf{U} = \{\xi \in \mathbb{C} \mid \xi^m = 1,\ \gcd(m,p) = 1\}$.
- Let us denote by $G^0$ the set of $p$-regular elements of $G$.

## Lemma 1

The restriction of $^*$ to $\mathbf{U}$ defines an isomorphism $\mathbf{U} \to F^\times$ of multiplicative groups. Also $F$ is an algebraically closed field of characteristic $p$.

## Brauer character

Suppose that $\mathcal{X} : G \to \mathrm{GL}(n, F)$ is a representation of $G$. If $g \in G^0$, by Lemma 1 we have that all the eigenvalues of $\mathcal{X}(g)$ (which lie in $F^\times$ since $F$ is algebraically closed and $\mathcal{X}(g)$ is an invertible matrix) are of the form $\xi_1^*, \ldots, \xi_n^*$ for uniquely determined $\xi_1, \ldots, \xi_n \in \mathbf{U}$. We say that $\varphi : G^0 \to \mathbb{C}$ defined for $g \in G^0$ by

$$\varphi(g) = \xi_1 + \cdots + \xi_n$$

is the **Brauer character** of $G$ afforded by the representation $\mathcal{X}$.

- We denote the set of irreducible Brauer characters of $G$ by $\mathrm{IBr}(G)$.
- $|\mathrm{IBr}(G)|$ equals the number of simple $FG$-modules.
- The next lemma tells us why Brauer characters are only defined on $p$-regular elements.

**Lemma 2**

If $\mathcal{X}$ is a representation of $FG$ with character $\psi$ and $g \in G$, then $\psi(g) = \psi(g_{p'})$. In particular, if $\mathcal{X}$ affords the Brauer character $\mu$, then $\psi(g) = \mu(g_{p'})^*$. Furthermore, if we define $\varphi^*(g) = \varphi(g_{p'})^*$ for $g \in G$ and $\varphi \in \mathrm{IBr}(G)$, then $\{\varphi^* | \varphi \in \mathrm{IBr}(G)\}$ is the set of the trace functions of the irreducible representations of $FG$.

- If $\chi \in \mathrm{Irr}(G)$, $\chi$ uniquely determines an algebra homomorphism $\omega_\chi : \mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$, given by

$$\omega_\chi(\widehat{K}) = \frac{|K|\chi(x_K)}{\chi(1)}.$$

- It is well known that $\omega_\chi(\widehat{K})$ is an algebraic integer.
- We may construct an algebra homomorphism $\lambda_\chi : \mathbf{Z}(FG) \to F$, given by

$$\lambda_\chi(\widehat{K}) = \omega_\chi(\widehat{K})^*.$$

- In the same way, if $\varphi \in \mathrm{IBr}(G)$, we can associate to $\varphi$ an algebra homomorphism $\lambda_\varphi : \mathbf{Z}(FG) \to F$.
- Let $\mathcal{X} : FG \to \mathrm{Mat}(n, F)$ be an irreducible $F$-representation of $G$ affording $\varphi$, then $\mathcal{X}(\widehat{K})$ is a scalar matrix, and this scalar only depends on $\varphi$. Hence, the equality

$$\mathcal{X}(\widehat{K}) = \lambda_\varphi(\widehat{K}) I_n$$

defines an algebra homomorphism $\lambda_\varphi : \mathbf{Z}(FG) \to F$.

- From the character theoretical point of view.

### $p$-block

The $p$**-blocks** of $G$ are the equivalence classes in $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ under the relation $\chi \sim \varphi$ if $\lambda_\chi = \lambda_\varphi$ for $\chi, \varphi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$.

- We denote the set of $p$-blocks of $G$ by $\mathrm{Bl}(G)$.
- If $B$ is a $p$-block of $G$, we write $\mathrm{Irr}(B) = B \cap \mathrm{Irr}(G)$ and $\mathrm{IBr}(B) = B \cap \mathrm{IBr}(G)$. If $\psi \in B$, we put $\lambda_B = \lambda_\psi$.
- R. Brauer defined an important invariant for a $p$-block, that is, every $p$-block $B$ of $G$ has canonically associated a $G$-conjugacy class of $p$-subgroups of $G$.

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Let $B \in \mathrm{Bl}(G)$ and $e_B = (\sum_{\chi \in \mathrm{Irr}(B)} e_\chi)^*$. Then $e_B \in \mathbf{Z}(FG)$, and so

$$e_B = \sum_{K \in \mathrm{cl}(G)} a_B(\widehat{K})\widehat{K},$$

where $a_B(\hat{K}) \in F$. Since

$$1 = \lambda_B(e_B) = \lambda_B(\sum_{K \in \mathrm{cl}(G)} a_B(\widehat{K})\widehat{K}) = \sum_{K \in \mathrm{cl}(G)} a_B(\widehat{K})\lambda_B(\widehat{K}),$$

there is at least one conjugacy class $K \in \mathrm{cl}(G)$ such that

$$a_B(\hat{K}) \neq 0 \neq \lambda_B(\hat{K}).$$

If this is the case, we call such a class a **defect class** for $B$.

- Let $K$ and $L$ be defect classes of $B$, and let $x_K \in K, x_L \in L$; if $P \in \mathrm{Syl}_p(\mathbf{C}_G(x_K))$ and $Q \in \mathrm{Syl}_p(\mathbf{C}_G(x_L))$, then $P$ and $Q$ are $G$-conjugate.

### defect group of a block

Let $B$ be a $p$-block of $G$ and let $K \in \mathrm{cl}(G)$ be a defect class of $B$. Let $x_K \in K$ and $P \in \mathrm{Syl}_p(\mathbf{C}_G(x_K))$. The **defect groups** of $B$ are in $\{P^g \mid g \in G\}$.

## defect of a block

If $B$ is a $p$-block of $G$, we define the **defect** $\mathrm{d}(B)$ of the block $B$ as the integer satisfying

$$p^{a-\mathrm{d}(B)} = \min\{\varphi(1)_p \mid \varphi \in \mathrm{IBr}(B)\},$$

where $|G|_p = p^a$. The analogous concept holds for irreducible ordinary characters.

## Theorem 3

If $D$ is a defect group of $B$, then $|D| = p^{\mathrm{d}(B)}$.

## height of a irreducible Brauer character

If $B$ is a block of $G$, $\varphi \in \mathrm{IBr}(B)$ and $|G|_p = p^a$, it follows that we may write

$$\varphi(1)_p = p^{a-\mathrm{d}(B)+h}$$

for a uniquely determined nonnegative integer $h$. We say that $h$ is the **height** of the irreducible Brauer character $\varphi$. The analogous concept holds for irreducible ordinary characters.

## Brauer Height Zero Conjecture(1955)

Let $B$ be a $p$-block of $G$ with defect group $D$. Then $\mathrm{ht}(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$ if and only if $D$ is abelian.

- For $p$-solvable groups
  - The "if" part of this conjecture was proved for $p$-solvable groups by Fong, and the "only if" part of this conjecture was proved for $p$-solvable groups by Gluck and Wolf.

P. Fong, On the characters of $p$-solvable groups, Trans. Amer. Math. Soc. 98 (1961) 263–284.

D. Gluck, T.R. Wolf, Brauer's height conjecture for $p$-solvable groups, Trans. Amer. Math. Soc. 282(1) (1984) 137–152.

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- Berger and Knörr showed that "if" direction of the conjecture holds for all groups, provided that it holds for all quasi-simple groups.

T.R. Berger, R. Knörr, On Brauer's height 0 conjecture, Nagoya Math. J. 109 (1988) 109–116.

- Navarro and Tiep proved both directions for 2-blocks of maximal defect.

G. Navarro, P.H. Tiep, Brauer's height zero conjecture for the 2-blocks of maximal defect, J. Reine Angew. Math. 669 (2012) 225–247.

- The knowledge of the blocks of quasisimple groups has been enough to complete a proof of "if" direction of the conjecture by Kessar and Malle, which relies on the result of Berger and Knörr.

R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, Ann. of Math.(2) 178(1) (2013) 321–384.

- Navarro and Späth were reduced “only if” direction of the conjecture to a question on simple groups.

G. Navarro, B. Späth, On Brauer’s height zero conjecture, J. Eur. Math. Soc. (JEMS) 16(4) (2014) 695–747.

- Malle and Navarro proved both directions for principal blocks for every prime.

G. Malle, G. Navarro, Brauer’s Height Zero Conjecture for principal blocks, J. Reine Angew. Math. 778 (2021) 119–125.

- “only if” direction of BHZC in the case that $p$ is odd and even

G. Malle, G. Navarro, M. Schaffer Fry, H.T. Tiep, Brauer’s Height Zero Conjecture, Ann. of Math.(2) 200(2) (2024) 557–608.

L. Ruhstorfer, The Alperin–McKay and Brauer’s height zero conjecture for the prime 2, to appear in Ann. of Math. (2024).

## Brauer Height Zero Conjecture(1955), now a Theorem

Let $B$ be a $p$-block of $G$ with defect group $D$. Then $\mathrm{ht}(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$ if and only if $D$ is abelian.


- R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, Ann. of Math.(2) 178(1) (2013) 321–384.
- G. Malle, G. Navarro, M. Schaffer Fry, H.T. Tiep, Brauer's Height Zero Conjecture, Ann. of Math.(2) 200(2) (2024) 557–608.
- L. Ruhstorfer, The Alperin–McKay and Brauer's height zero conjecture for the prime 2, to appear in Ann. of Math. (2024).

- Write $\mathrm{mh}(B)$ for the minimal nonzero height of an irreducible character in $B$, and similarly (by a slight abuse of notation) $\mathrm{mh}(D)$ for the minimal nonzero height of any irreducible character of $D$. We set $\mathrm{mh}(B) = \infty$ (and similarly $mh(D) = \infty$) if there is no irreducible character of positive height, and use the convention that $n < \infty$ for all $n \in \mathbb{Z}$.
- Brauer's height zero conjecture is equivalent to $\mathrm{mh}(B) = \infty$ if and only if $\mathrm{mh}(D) = \infty$.

### Eaton–Moretó Conjecture(2014)

Let $B$ be a $p$-block of a finite group $G$ with defect group $D$. Then $\mathrm{mh}(B) = \mathrm{mh}(D)$.

C. Eaton, A. Moretó, Extending Brauer's height zero conjecture to blocks with nonabelian defect groups, Int. Math. Res. Not. (IMRN) 2014 (2014) 5581–5601.

- If $G$ is $p$-solvable, then $\mathrm{mh}(D) \leq \mathrm{mh}(B)$. (Eaton–Moretó)
- If $B$ is the principal $p$-block of $G$ and $D$ has two character degrees, then $\mathrm{mh}(B) \leq \mathrm{mh}(D)$. (Malle–Moretó–Rizo)
- There is a version of this conjecture for fusion systems. (Kessar–Linckelmann–Lynd–Semeraro)

- Eaton–Moretó Conjecture has recently been proved to be true for
  - General Linear Groups in characteristic $p$, Symmetric groups, Sporadic Groups (Eaton–Moretó)
  - 2-blocks with minimal nonabelian defect groups (Eaton–Külshammer–Sambale)
  - 2-blocks of defect at most 4 (Külshammer–Sambale)
  - the principal block of any quasi-simple group, all $p$-blocks of quasi-simple groups of Lie type in characteristic $p$, all unipotent blocks of quasi-simple exceptional groups of Lie type, all $p$-blocks of covering groups of an alternating or symmetric group (Brunat–Malle)
  - all blocks of finite General Linear and Unitary Groups for all primes, all blocks of finite quasi-simple groups of type A (Feng–Liu–Zhang)
  - principal $p$-block of a $p$-solvable group (Navarro)

### Navarro

It is natural to ask if a reduction of the conjecture to a question on simple groups is possible, at least in the principal block case. (difficulty!!!)

- Our first aim: prove the general $p$-solvable case of the conjecture. (work in progress)

- If $H$ is a subgroup of $G$ and $b \in \mathrm{Bl}(H)$, we extend the algebra homomorphism $\lambda_b : \mathbf{Z}(FH) \to F$ to a linear map $\lambda_b^G : \mathbf{Z}(FG) \to F$ by setting
$$\lambda_b^G(\widehat{K}) = \lambda_b(\sum_{x \in K \cap H} x),$$
where it is understood that the sum $\sum_{x \in K \cap H} x$ is zero if $K \cap H$ is empty.
- It may happen that $\lambda_b^G$ is an algebra homomorphism. If this is the case, there exists a unique block $b^G \in \mathrm{Bl}(G)$ such that $\lambda_b^G = \lambda_{b^G}$. We will say in this case that $b^G$ is **defined**. The block $b^G$ is called the **induced block**.

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# global $\longleftrightarrow$ local

## Brauer's first main theorem

Block induction defines a bijection between the blocks of $\mathbf{N}_G(D)$ with defect group $D$ and the blocks of $G$ with defect group $D$.

- In the situation of Brauer's first main theorem, we say $b$ and $b^G$ are **Brauer correspondent** blocks.

## McKay Conjecture(1971)

Let $P$ be a Sylow $p$-subgroup of $G$. Then $|\operatorname{Irr}_{p'}(G)| = |\operatorname{Irr}_{p'}(\mathbf{N}_G(P))|$, i.e. there is a bijection $f : \operatorname{Irr}_{p'}(G) \to \operatorname{Irr}_{p'}(\mathbf{N}_G(P))$.

- Alperin introduced an important generalisation of the McKay Conjecture by considering $p$-blocks.

## Alperin–McKay Conjecture(1975)

Let $B$ be a $p$-block of $G$ and let $b$ be the Brauer correspondent of $B$. Then $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}_0(b)|$, i.e. there is a bijection $g : \operatorname{Irr}_0(B) \to \operatorname{Irr}_0(b)$.

- Okuyama proved the modular analogue of the Alperin–McKay Conjecture for $p$-solvable groups.

- There has been some recent interest in **finding bijections with additional properties**.
- For instance,
  - Navarro Conjecture proposes that there should be $f$ that commutes with the $\mathcal{G}_p$-action, where $\mathcal{G}_p$ consists of those $\sigma \in \mathcal{G} = \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ that send every root of unity $\xi \in \mathbb{Q}_{|G|}$ of order not divisible by p to $\xi^{p^e}$, where $p^e$ is a fixed but arbitrary power of $p$. Also for Brauer correspondence blocks.
  - the Isaacs–Navarro Conjecture establishes that there should be $f$ such that $f(\chi)(1) \equiv \pm\chi(1) \pmod{p}$ for $\chi \in \mathrm{Irr}_{p'}(G)$.

- For instance,
  - For a solvable group $G$, Turull showed that there should be $f$ such that $f(\chi)(1) \mid \chi(1)$ for $\chi \in \mathrm{Irr}_{p'}(G)$.
  - Rizo extended Turull's result to $p$-solvable groups by assuming the fact that the Glauberman correspondence is compatible with divisibility of character degrees, the version for Brauer characters obtained by Bonazzi, Navarro, Rizo and Sanus.
  - For a $p$-solvable group $G$, Martínez and Rossi showed that there should be $g$ such that $g(\chi)(1) \mid \chi(1)$ for $\chi \in \mathrm{Irr}_0(B)$, also holds for Brauer characters.
  - $\cdots$

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- If $N \trianglelefteq G$, $b$ is a block of $N$, and $g \in G$, the set $\{\psi^g \mid \psi \in \mathrm{Irr}(b) \cup \mathrm{IBr}(b)\}$ is a block of $N$, namely $b^g$. Hence $G$ acts on $\mathrm{Bl}(N)$ by conjugation. If $\{b_1, \ldots, b_t\}$ is the $G$-orbit of $b$, we have that the idempotent $\sum_{i=1}^{t} e_{b_i}$ lies in $\mathbf{Z}(FG)$ and there exist uniquely determined blocks $B_1, \ldots, B_s \in \mathrm{Bl}(G)$ such that
$$\sum_{i=1}^{t} e_{b_i} = \sum_{i=1}^{s} e_{B_i}.$$
- We say in this case that the block $B_i$ **covers** $b$. We write $\mathrm{Bl}(G|b) = \{B_1, \ldots, B_s\}$ for the set of blocks of $G$ which cover $b$.

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## Fong–Reynolds Theorem

Let $N \trianglelefteq G$ and let $b \in \mathrm{Bl}(N)$. Let $T(b)$ be the stabilizer of $b$ in $G$.

(a) The map $\mathrm{Bl}(T(b)|b) \to \mathrm{Bl}(G|b)$ given by $B \mapsto B^G$ is a bijection;

(b) If $B \in \mathrm{Bl}(T(b)|b)$, then $\mathrm{Irr}(B^G) = \{\psi^G \mid \psi \in \mathrm{Irr}(B)\}$ and $\mathrm{IBr}(B^G) = \{\varphi^G \mid \varphi \in \mathrm{IBr}(B)\}$;

(c) If $B \in \mathrm{Bl}(T(b)|b)$, then every defect group of $B$ is a defect group of $B^G$;

(d) If $B \in \mathrm{Bl}(T(b)|b)$ and $\chi \in \mathrm{Irr}(B)$, then $\mathrm{ht}(\chi) = \mathrm{ht}(\chi^G)$.

- In the situation of Fong–Reynolds theorem, we say $B$ and $B^G$ are **Fong–Reynolds correspondent** blocks.

## Harris–Knörr Theorem

Let $N$ be a normal subgroup of a finite group $G$, $b$ be a $p$-block of $N$ with defect group $D$. Let $b^*$ be the Brauer first main correspondent of $b$ in $\mathbf{N}_N(D)$. Then block induction defines a defect group preserving bijection

$$\mathrm{Bl}(\mathbf{N}_G(D)|b^*) \to \mathrm{Bl}(G|b), B \mapsto B^G.$$

- In the situation of Harris–Knörr theorem, we say $B$ and $B^G$ are **Harris–Knörr correspondent** blocks.

Let $G$ be a $p$-solvable group. Let $B$ and $B^G$ be Harris–Knörr correspondent blocks.

- (Laradji) $B$ and $B^G$ contain equal numbers of irreducible Brauer characters over height zero irreducible Brauer characters.
- (Laradji) If $G$ is of odd order, then there exists a height preserving bijection from the set of irreducible complex characters of $B$ lying over height zero characters onto the set of irreducible complex characters of $B^G$ lying over height zero characters.
- (Navarro and Späth) Without the assumption that "$G$ is of odd order".

## Our Work

- Following Navarro and Späth, we considered the question.

**Question**

Let $N$ be a normal subgroup of a finite $p$-solvable group $G$, and let $B$ and $b$ be respectively $p$-blocks of $G$ and $N$ such that $B$ covers $b$. Assume that $D$ is a defect group of $b$. Let $b^*$ be the Brauer first main correspondent of $b$ in $\mathbf{N}_N(D)$, and let $B^* \in \mathrm{Bl}(\mathbf{N}_G(D)|b^*)$ be the Harris–Knörr correspondent of $B$. Is there a bijection $\Pi_D : \mathrm{IBr}_0(b) \to \mathrm{IBr}_0(b^*)$ and a height preserving bijection from $\mathrm{IBr}(B|\tau)$ onto $\mathrm{IBr}(B^*|\Pi_D(\tau))$ for every $\tau \in \mathrm{IBr}_0(b)$?

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## Our Work

- a version for Brauer characters of Navarro and Späth.

**Theorem(G. and Liu)**

Let $N$ be a normal subgroup of a finite $p$-solvable group $G$, and let $B$ and $b$ be respectively $p$-blocks of $G$ and $N$ such that $B$ covers $b$. Assume that $D$ is a defect group of $b$. Let $b^*$ be the Brauer first main correspondent of $b$ in $\mathbf{N}_N(D)$, and let $B^* \in \mathrm{Bl}(\mathbf{N}_G(D)|b^*)$ be the Harris–Knörr correspondent of $B$. Suppose that every irreducible Brauer character in $\mathrm{IBr}(b)$ is of height zero. Then there is a height preserving bijection from $\mathrm{IBr}(B)$ onto $\mathrm{IBr}(B^*)$.

# Thanks for your attention!