VERTEX-TRANSITIVE GRAPHS AND DERANGEMENTS

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Question (Korchmáros – 2022)

Let Γ be a finite d-valent graph, and let G be a vertex-transitive group of automorphisms of Γ. Does there exists a function $\mathbf{f} : \mathbb{N} \to (0,1)$ such that

$$\delta(G \circlearrowright V\Gamma) \geq \mathbf{f}(d)?$$

Let G be a finite and transitive permutation group on Ω.

For every $x \in G$, let

$$\pi(x) := \#\text{ points fixed by } x$$

be the **permutation character of $G \circlearrowright \Omega$**.

A **derangement** is a permutation $x \in G$ without any fixed points, that is,

$$\pi(x) = 0.$$

We define the **portion of derangements of G** as the ratio

$$\delta(G \curvearrowright \Omega) = \frac{\#\text{ derangements}}{|G|}.$$

Example A. Suppose that G is regular on Ω. Then

$$\delta(G \curvearrowright \Omega) = \frac{|G| - 1}{|G|}.$$

Example B. Using the inclusion–exclusion principle,

$$\delta(\mathrm{Sym}(n) \curvearrowright \{1, 2, \ldots, n\}) = \sum_{k=2}^{n} \frac{(-1)^k}{k}.$$

Not Burnside's Lemma (or Orbit Counting Lemma)

Let $G \circlearrowright \Omega$ be a transitive permutation group. Then

$$|G| = \sum_{g \in G} \pi(g).$$

Theorem 1 (Jordan – 1872)

Every nontrivial transitive permutation group contains a derangements, that is,

$$\delta(G \circlearrowright \Omega) > 0.$$

Theorem 2 (Serre – 2003)

- Let $f \in \mathbb{Z}[t]$ be an irreducible polynomial over $\mathbb{Q}$ of degree $n \geq 2$. Then f has no roots modulo p for infinitely many primes p.
- Let $\varphi : T \to S$ be a covering map of topological spaces of degree $n \geq 2$. Then there is a continuous closed curve in S which cannot be lifted to T.

A **semiregular element** is a derangement whose nontrivial powers are also derangements, that is, for every integer h that does not divide $\mathbf{o}(x)$,

$$\pi\left(x^h\right) = 0.$$

Polycirculant Conjecture (Marušič – 1981)

> The automorphism group of any vertex-transitive graph contains a semiregular permutation.

This conjecture sparked interest in **elusive permutation groups**, that is, those permutation group that contain no semiregular elements. But this is another story ....

Theorem 3 (Cameron, Kovács, Newman, Praeger – 1985)

Let G be a p-group acting transitively on Ω. Then

$$\delta(G \curvearrowright \Omega) > \frac{p-1}{p+1}.$$

Theorem 4 (Fulman, Guralnick – 2003-2018)

There exists a universal constant $\epsilon > 0$ such that, for every simple group G acting transitively on Ω,

$$\delta(G \curvearrowright \Omega) > \epsilon.$$

Example C. Let p be an odd prime, and let $G = \mathrm{P\Gamma L}_2(2^p)$ be a permutation group with stabilizer $H = C_{2^p+1} \rtimes C_{2p}$.

We can check that all the derangements of this action are trapped in $\mathrm{PGL}_2(2^p)$.

Therefore,

$$\lim_p \delta(G \curvearrowright G/H) \leq \lim_p \frac{1}{p} = 0.$$

Let G_α be the stabilizer of the point $\alpha \in \Omega$.

We call **suborbits** the G_α-orbits on Ω, say

$$O_1 = \{\alpha\}, O_2, \ldots, O_r.$$

The integer r is the **permutation rank** of $G \curvearrowright \Omega$.

Example D. If $G \curvearrowright \Omega$ is 2-transitive, then $r = 2$.

Theorem 5 (Cameron, Cohen – 1992)

Let $G \curvearrowright \Omega$ be a transitive permutation group of degree n and permutational rank r. Then

$$\delta(G \curvearrowright \Omega) \geq \frac{r-1}{n}.$$

Equality is attained if, and only if, G is a Frobenius group.

The **subdegrees** of $G \circlearrowleft \Omega$ are the length of its suborbits, that is,

$$d_k = |O_k|.$$

Example E. For every permutation group, $d_1 = 1$.

We call **minimal nontrivial subdegrees** the integer

$$d := \min\{d_2, d_3, \dots, d_r\}.$$

Example F. Let G be the group of rotations of the cube Q_8. Note that, if a rotation fixes a vertex of the cube, then it also fixes its antipodal. Hence, $d = 1$.

An **orbital graph for $G \circlearrowright \Omega$** is a (simple finite) graph Γ such that

$$(V\Gamma, E\Gamma) = \left(\Omega, \{\alpha, \beta\}^G\right).$$

Example G. For every permutation group $G \circlearrowright \Omega$,

$$\left(\Omega, \mathrm{diag}(\Omega^2) = \{(\alpha, \alpha) \mid \alpha \in \Omega\}\right)$$

is an orbital graph, which we call **diagonal orbital graph**.

Example H. The orbital graphs of $\mathrm{Sym}(5)$ acting on 2-subsets are the diagonal orbital graph on 10 vertices, the Petersen graph, and its complement.

Lemma 6

The mapping

$$\varphi_\alpha : \beta^{G_\alpha} \mapsto \left(\Omega, \{\alpha, \beta\}^G\right)$$

is a one-to-one correspondence between suborbits and orbital graphs.

In particular,

- the permutation rank r of $G \curvearrowright \Omega$ is the number of distinct orbital graphs for the action;
- the minimal nontrivial subdegree d is the minimal valency of a (meaningful) graph on which G acts transitively;
- all the graphs on which G acts are union of suitable orbital graphs.

Refined Question

Does there exist a function $\mathbf{f} : \mathbb{N} \to (0, 1)$ such that, for every transitive permutation group $G \curvearrowright \Omega$ of minimal nontrivial subdegree d,

$$\delta(G \curvearrowright \Omega) \geq \mathbf{f}(d)?$$

Is the minimal nontrivial subdegree a **natural parameter to study for a permutation group**?

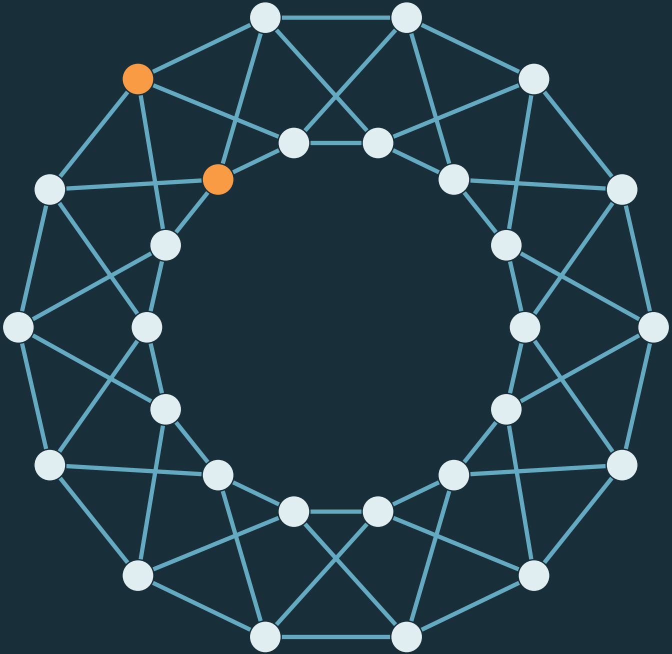
Theorem 7 (Cameron, Praeger, Saxl, and Seitz – 1983)

There exists a function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that, for every primitive permutation group $G \curvearrowright \Omega$ of minimal nontrivial subdegree d, and for every $\alpha \in \Omega$,

$$|G_\alpha| \leq \mathbf{f}(d).$$

Many generalization of Theorem 7 has been conjectured, the most influential being Weiss Conjecture.

On the other hand, it is well known where similar result must fail: for the infinite families of Praeger–Xu graphs.



Lemma 8

Let G be an arc-transitive group of automorphisms of the connected d-valent graph Γ. The pair (Γ, G) has a universal cover of the form

$$\left(\mathcal{T}_d, G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}\right),$$

where α and β are two adjacent vertices of Γ, and $\mathcal{T}_d$ is the infinite tree of valency d.

Still, one can check that

$$\exp(G_\alpha) \quad \text{and} \quad \mathbf{d}\left(G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}\right)$$

are uniformly bounded for the family of Praeger–Xu graphs.

Is this true in general?

Theorem 9 (B., Spiga – 2024†)

There exists no function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that, for every G acting arc-transitively on a connected d-valent graph Γ,

$$\mathbf{d}(G) \leq \mathbf{f}(d).$$

Question 10 (Praeger, Pyber, Spiga, Szabó – 2012)

Is there a function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that, for every G acting arc-transitively on a connected d-valent graph Γ, and for every vertex $\alpha \in V\Gamma$,

$$\exp(G_\alpha) \leq \mathbf{f}(d)?$$

Back to our. . . .

Refined Question

Does there exist a function $\mathbf{f} : \mathbb{N} \to (0, 1)$ such that, for every transitive permutation group $G \circlearrowright \Omega$ of minimal nontrivial subdegree d,

$$\delta(G \circlearrowright \Omega) \geq \mathbf{f}(d)?$$

Define

$$F_h := \#\{g \in G \mid \pi(g) = h\},$$

and note that

$$|G| = \sum_{h=0}^{n} F_h \quad \text{and} \quad |G| = \sum_{h=1}^{n} hF_h\,.$$

We compute

$$F_0 = \sum_{h=2}^{n} (h-1)F_h \geq \sum_{h=2}^{n} F_h = |G| - F_0 - F_1\,.$$

By rearranging the terms, we get

$$2F_0 = |G| - F_1\,.$$

We define, for every $\omega \in \Omega$

$$F_1^{\omega} := \left| G_{\omega} \cap \pi^{-1}(1) \right|.$$

By transitivity,

$$F_1 = \sum_{\omega \in \Omega} F_1^{\omega} = n F_1^{\alpha}.$$

By choosing β in a G_{α}-orbit of minimal length, we obtain

$$F_1^{\alpha} \leq |G_{\alpha}| - |G_{\alpha\beta}| = |G_{\alpha}| \left(1 - \frac{1}{d} \right).$$

We substitute in the last equation of the previous slide,

$$2F_0 \geq |G| - n|G_{\alpha}| \left(1 - \frac{1}{d} \right) = |G| \left(1 - 1 + \frac{1}{d} \right).$$

Theorem 11 (B., Spiga – 2024)

Let G be a transitive group on Ω of degree n and minimal nontrivial subdegree d. Then

$$\delta(G \circlearrowright \Omega) \geq \frac{1}{2d} + \frac{n-2}{2|G|}.$$

Equality is attained if, and only if G is a Frobenius group.

Corollary 12 (B., Spiga – 2024)

Yes, take the function

$$\mathbf{f}(d) = \frac{1}{2d}.$$

Is our bound better than Cameron–Cohen bound?

Example I. Let G be a non-Frobenius 2-transitive group of degree n, so that

$$|G| > n(n-1).$$

We have

$$\frac{1}{2d} + \frac{n-2}{2|G|} < \frac{1}{2(n-1)} + \frac{n-2}{2n(n-1)} = \frac{1}{n}.$$

Is our bound better than Cameron–Cohen bound?

Example L. Let $G = \mathrm{PSL}_2(p)$ with $p = 43 \pmod{120}$ a prime, and let $H = \mathrm{Alt}(4)$. Consider the action of G on G/H by multiplication. The minimal nontrivial subdegree is 4, while asymptotically the rank is

$$r = \frac{p^3}{24 \cdot 12} + O(p).$$

Therefore,

$$\lim_p \frac{r-1}{n} = \frac{1}{12} < \frac{1}{8} = \frac{1}{2d}.$$

Two questions to think about when you are bored.

- Can you create an oracle that, given a finite permutation group, without any computation, tells us whether our bound or the Cameron–Cohen bound is better?
- Can you find some application to the Corollary 12?