Weakly distance-regular digraphs with P-polynomial property

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Outline

1. Introduction
   - Digraph
   - Association scheme
   - Distance-regular graphs
   - Distance-regular digraph
   - Weakly distance-regular digraphs
2. Development on wdrdgs
3. Our works
   - P-polynomial case
   - Underlying graphs are Hamming graphs or related graphs
   - Underlying graphs are Johnson graphs or related graphs

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Digraph

- A digraph Γ is a pair $(V(\Gamma), A(\Gamma))$ where $V(\Gamma)$ is a finite set of vertices and $A(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ is a set of arcs.
- Γ is an undirected graph or a graph if $A(\Gamma)$ is a symmetric relation.

Path

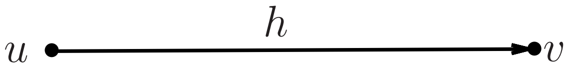
- A path from u to v:

$$(u = w_0, w_1, \ldots, w_r = v)$$

  such that (w_{t-1}, w_t) is an arc for $t = 1, 2, \ldots, r$.
- r is the length of the path (the number of arcs).

Distance and Diameter

- Distance $\partial(u, v)$: the length of a shortest path from u to v.
- If $\partial(u, v) = h$, write



- The maximum value of distance function is the diameter of Γ.

Circuit

- The path $(w_0, w_1, \ldots, w_{r-1})$ is a circuit if (w_{r-1}, w_0) is an arc.
- The girth of Γ is the length of a shortest circuit in Γ.

Strongly connected

- A digraph (resp. graph) Γ is said to be strongly connected (resp. connected) if, for any two vertices x and y, there is a path from x to y.

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Association scheme

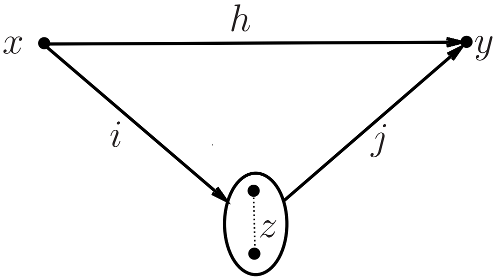
- X: finite set.
- $R_0, R_1, \ldots, R_d$: relations, nonempty subset of $X \times X$.
- Write ${}^tR_i = \{(y,x) \mid (x,y) \in R_i\}$.
- $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ is called an association scheme with d classes, if

Association scheme

(i) $R_0 = \{(x,x) \mid x \in X\}$;
(ii) $R_0, R_1, \ldots, R_d$ is a partition of $X \times X$;
(iii) ${}^tR_i = R_{i'}$ for some $i' \in \{0,1,\ldots,d\}$;
(iv) For any $(x,y) \in R_h$,

$$p^h_{i,j} = |\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}|$$

depends only on i, j, h. (intersection numbers!)



Association scheme

- $\mathfrak{X}$ is commutative if $p_{i,j}^h = p_{j,i}^h, \ \forall\ i, j, h$.
- $\mathfrak{X}$ is symmetric if each R_i is symmetric ($R_i = {}^tR_i,\ \forall i$) and non-symmetric otherwise.

Adjacency matrix

- For each R_i, let A_i be the binary matrix, whose rows and columns are indexed by the elements of X such that

$$(A_i)_{xy} = 1 \Longleftrightarrow (x, y) \in R_i.$$

- A_i is called ith adjacency matrix of $\mathfrak{X}$.

Main topic of AS

- Main topic of AS: Classify all AS.
- BUT AS are too general, it is impossible!
- Which families of AS are important?

Outline

P-polynomial scheme

- An association scheme $(X, \{R_i\}_{0\leq i\leq d})$ is P-polynomial with respect to the ordering $R_0, R_1, \ldots, R_d$, if for each i, there exists a complex coefficient polynomial $v_i(x)$ of degree i such that
$$A_i = v_i(A_1).$$

Distance-regular graphs

- Let $(X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric P-polynomial association scheme with respect to the ordering $R_0,\ R_1, \ldots, R_d$. Then (X, R_1) is a connected graph, called a distance-regular graph.

- In this talk, a graph means a connected graph.

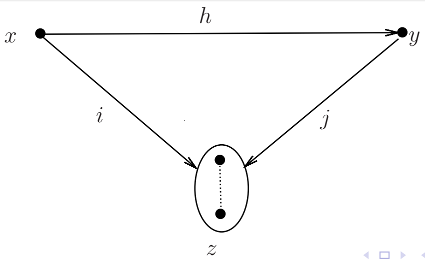
Definition of DRG

Definition (Biggs, 70's)

A graph Γ is said to be distance-regular (DRG), if for x, y with $\partial(x,y) = h$,

$$p^h_{i,j} = |\{z \in V\Gamma \mid \partial(x,z) = i, \partial(y,z) = j\}|$$

depends only on i, j, h.



Remarks

- Given a DRG of diameter d, let

$$R_i = \{(x, y) \in X \times X \mid \partial(x, y) = i\}.$$

  Then $(X, \{R_i\}_{0 \leq i \leq d})$ is a symmetric P-polynomial AS with respect to the ordering $R_0,\ R_1, \ldots, R_d$.
- Symmetric P-polynomial AS $\Longleftrightarrow$ DRG.

Question

- Recall: A symmetric P-polynomial AS determines a DRG.
- How about non-symmetric case?

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Distance-regular digraphs

- Let $(X, \{R_i\}_{0\leq i\leq d})$ be a non-symmetric P-polynomial AS with respect to the $R_0,\ R_1, \ldots, R_d$. The (X, R_1) is a strongly connected digraph, is called a distance-regular digraph.
- In this talk, a digraph means a strongly connected digraph, not undirected.

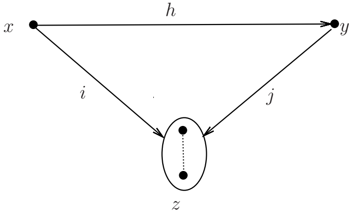
Definition of DRDG

Definition (Damerell, 1981)

A digraph Γ is said to be distance-regular (DRDG) , if for x, y with $\partial(x,y)=h$,

$$p^h_{i,j}=|\{z\in V\Gamma \mid \partial(x,z)=i, \partial(y,z)=j\}|$$

depends only on i, j, h.



R.M. Damerell, Distance-transitive and distance-regular digraph, J. Combin. Theory Ser.B, 31 (1981), 46–53.

Remarks

- Note that the unique difference between the definitions of DRDG and DRG:
  In DRDG, "digraph"; In DRG, "graph".
- Given a DRDG of diameter d, let

$$R_i = \{(x, y) \mid \partial(x, y) = i\}.$$

  Then $(X, \{R_i\}_{0 \leq i \leq d})$ be a non-symmetric P-polynomial AS.
- Non-symmetric P-polynomial AS $\Longleftrightarrow$ DRDG.

Distance-regular digraphs

- In 1981, Damerell proved that the diameter d of DRDG is $g-1$ (short) or g (long). Moreover, a long DRDG is a coclique extension of a short DRDG.
- In 1993, Leonard and Nomura proved that except directed cycles all short DRDG have $d=g-1\leq 7$.
- Since then, there has been very little progress. In fact, there are very few examples of DRDG.

R.M. Damerell, Distance-transitive and distance-regular digraph, J. Combin. Theory Ser.B, 31 (1981), 46–53.

D.A. Leonard and K. Nomura, the girth of a directed distance-regular graph, J. Combin. Theory Ser. B 58 (1993), 34–39.

Directed version

- A natural directed version of DRG with unbounded girth.

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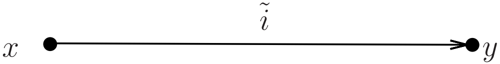
Two way distance

- In a graph, $\partial(x, y) = \partial(y, x)$.
- In a digraph, it does not hold. In order to describe the distance between two vertices x and y in a digraph:
- Two way distance $\tilde{\partial}(x, y) = (\partial(x, y), \partial(y, x))$.
- If $\tilde{\partial}(x, y) = \tilde{i}$, write



Two way distance

- $\tilde{\partial}(\Gamma)$: the set of all pairs $\tilde{\partial}(x,y)$.
- $\Gamma_{\tilde{i}}$: the set of ordered pairs (x,y) with $\tilde{\partial}(x,y)=\tilde{i}$, where $\tilde{i}\in\tilde{\partial}(\Gamma)$.

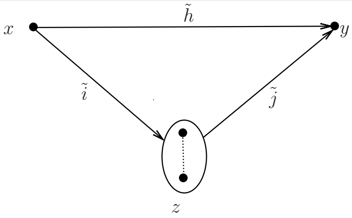
Wdrdg

Definition (Wang and Suzuki, 2003)

A digraph Γ is said to be weakly distance-regular (wdrdg) if, for any $\tilde{\partial}(x,y)=\tilde{h}$,

$$p^{\tilde{h}}_{\tilde{i},\tilde{j}}=|\{z\in V\Gamma\mid\tilde{\partial}(x,z)=\tilde{i}\text{ and }\tilde{\partial}(z,y)=\tilde{j}\}|$$

depends only on $\tilde{i},\tilde{j},\tilde{h}$.



K. Wang and H. Suzuki, Weakly distance-regular digraphs, Discrete Math. 264 (2003) 225-236.

The attached scheme

- $\mathfrak{X}(\Gamma) = (V\Gamma, \{\Gamma_{\tilde{i}}\}_{\tilde{i} \in \tilde{\partial}(\Gamma)})$ is an association scheme. We call $\mathfrak{X}(\Gamma)$ the attached scheme of Γ.
- $p^{\tilde{h}}_{\tilde{i},\tilde{j}}$: intersection number.
- Γ is commutative if the attached scheme is commutative.

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Development of wdrdgs

- Small valency.
- Small intersection numbers.
- Large intersection numbers.
- Circulant case.
- Locally semicomplete case.

Small valency: Valency 2

Theorem (Wang and Suzuki 2003)

A commutative wdrdg of valency 2 is isomorphic to one of the following Cayley digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_3^2, \{(0,1),(1,0)\})$.

(2) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1,2\})$.

(3) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1,n+1\})$.

(4) $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_n, \{(0,1),(1,0)\})$.

In 2004, Suzuki proved the nonexistence of noncommutative 2-valent wdrdgs.

K. Wang and H. Suzuki, Weakly distance-regular digraphs, Discrete Math. 264 (2003) 225-236.

H. Suzuki, Thin weakly distance-regular digraphs, J. Combin. Theory Ser. B 92 (2004), 69-83.

Small valency: Valency 3

- In 2004, Wang classified wdrdg of valency 3 and girth 2.
- Suzuki, Yang, Lv and Wang classified commutative wdrdg of valency 3 and girth more than 2.

H. Suzuki, Thin weakly distance-regular digraphs, J. Combin. Theory Ser. B 92 (2004), 69-83.

K. Wang, Commutative weakly distance-regular digraphs of girth 2, Europ. J. Combin., 25(2004), 363-375.

Y. Yang, B. Lv and K. Wang, Weakly distance-regular digraphs of valency three, I, Electron. J. Combin., 23(2) (2016) Paper 2.12.

Y. Yang, B. Lv and K. Wang, Weakly distance-regular digraphs of valency three, II, J. Combin. Theory Ser. A, 160 (2018) 288–315.

Small intersection numbers: thin

A wdrdg is thin if all the intersection numbers are at most 1.

Theorem (Suzuki 2004)

A thin wdrdg is isomorphic to one of the following Cayley digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_n, \{1\})$.

(2) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1, 2\})$.

(3) $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_n, \{(1, 0), (0, 1)\})$.

(4) $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2n}, \{(1, 0), (0, 1), (0, 2)\})$.

H. Suzuki, Thin weakly distance-regular digraphs, J. Combin. Theory Ser. B 92 (2004), 69-83.

Small intersection numbers: quasi-thin

A wdrdg is said to be quasi-thin if the maximum value of all intersection numbers is 2.

Small intersection numbers: quasi-thin

Theorem (Yang, Lv and Wang, 2020)

If Γ is a commutative quasi-thin wdrdg of valency more than 3 and vertices more than 16, then Γ is isomorphic to:

(1) $\mathrm{Cay}(\mathbb{Z}_{4p}, \{1, 2, 2p+i, 2p+1, 2p+2\})$, $p \neq 2-i$.
(2) $\mathrm{Cay}(\mathbb{Z}_q \times \mathbb{Z}_4, \{(0,1), (1,0), (1,2), (0, 2+i)\})$, $q \neq 3+i$.
(3) $\mathrm{Cay}(\mathbb{Z}_{2q} \times \mathbb{Z}_2, \{(0,1), (1,0), (2,0), (1,1)\})$.
(4) $\mathrm{Cay}(\mathbb{Z}_{4q} \times \mathbb{Z}_2, \{(0,1), (1,0), (2,0), (2q+1,0), (2q+2,0), (2qi,1)\})$, $q \notin \{3, 3+i\}$.
(5) $\mathrm{Cay}(\mathbb{Z}_{2q} \times \mathbb{Z}_4, \{(0,1), (1,0), (1,2), (0,2-i), (2,0), (2,2)\})$, $q \notin \{3, 3+i\}$.
(6) $\mathrm{Cay}(\mathbb{Z}_{2q} \times \mathbb{Z}_n, \{(0,1), (1,0), (2,0), (0,-1)\})$.
(7) $\mathrm{Cay}(\mathbb{Z}_{2q} \times \mathbb{Z}_n, \{(0,1), (1,(c+1)/2), (1,(c-1)/2), (2,c), (0,-1)\})$.
(8) $\mathrm{Cay}(\mathbb{Z}_{2n} \times \mathbb{Z}_q, \{(0,1), (1,(t+1)/2), (-1,(1-t)/2), (2,t), (-2,-t)\})$.

Here, $i \in \{0,1\}$, $2 \le p$, $3 \le q$, $3 \le n \le q - (1+(-1)^q)/2$, $c = n/\gcd(q,n)$, $t = q/\gcd(q,n)$ and c, t are both odd.

Y. Yang, B. Lv and K. Wang, Quasi-thin weakly distance-regular digraphs, J.

Large intersection numbers: thick

- A wdrdg is said to be thick if the two families of intersection numbers

$$p^{(h_1,h_2)}_{(i_1,i_2),(i_1,i_2)}, p^{(h_1,h_2)}_{(i_1,i_2),(i_2,i_1)}$$

are zero, or reach the maximum.

Thick, reduced theorem

Theorem (Yang and Wang, 2022)

Let Γ be a communicate thick wdrdg. Then there exsits a subdigraph Δ (thick wdrdg) s.t. Γ/Δ is isomorphic to thick wdrdgs:

(1) $\mathrm{Cay}(\mathbb{Z}_p, \{1\})$;

(2) $\mathrm{Cay}(\mathbb{Z}_p, \{1\})[\overline{K}_2]$;

(3) $\mathrm{Cay}(\mathbb{Z}_{2q-2}, \{1, 2\})$;

(4) $\mathrm{Cay}(\mathbb{Z}_{2q-2}, \{1, 2\})[\overline{K}_2]$;

(5) $\mathrm{Cay}(\mathbb{Z}_\gamma \times \mathbb{Z}_\eta, \{(2^\alpha + \beta, 1), (2^\alpha - \beta, \alpha), (2^{\alpha+1}, \alpha + 1)\})$;

(6) $\mathrm{Cay}(\mathbb{Z}_\gamma \times \mathbb{Z}_\eta, \{(2^\alpha + \beta, 1), (2^\alpha - \beta, \alpha), (2^{\alpha+1}, \alpha + 1)\})[\overline{K}_2]$.

Here, $q, p, \alpha, \beta, \gamma, \eta$......

Y. Yang and K. Wang, Thick weakly distance-regular digraphs, Graphs Combin., 38 (2022) Paper No. 37.

Circulant case

A circulant is a Cayley digraph of a cyclic group.

Theorem (Munemasa, W, Yang, Zhu, 2024^+)

A weakly distance-regular circulant with one type of arcs is isomorphic to

(1) $C_l[\overline{K}_m]$;

(2) $P(q)[\overline{K}_m]$;

(3) $C_3 \times K_h$;

(4) $\mathrm{Cay}(\mathbb{Z}_{13}, \{1, 3, 9\})[\overline{K}_m]$.

Here, $P(q)$ is the Paley tournament of order prime q, $q \equiv 3 \pmod 4$, $m \geq 1$, $l \geq 3$, $h, q > 3$, $3 \nmid h$.

A. Munemasa, K. Wang, Y. Yang and W. Zhu, Weakly distance-regular circulants, I, arXiv:2307.12710.

Locally semicomplete case

- A digraph Γ is semicomplete, if for any pair of vertices $x, y \in V(\Gamma)$, either $(x, y) \in A(\Gamma)$, or $(y, x) \in A(\Gamma)$, or both.
- A digraph Γ is locally semicomplete, if $\Gamma[N^+(x)]$ and $\Gamma[N^-(x)]$ are both semicomplete for every vertex x of Γ.

Locally semicomplete case

Theorem (Yang, Li and Wang, 2024^+)

Let Γ be a commutative weakly distance-regular digraph of valency more than 3. Then Γ is locally semicomplete but not semicomplete if and only if Γ is isomorphic to one of the following digraphs:

(1) $\Lambda \circ K_m$;

(2) $\mathrm{Cay}(\mathbb{Z}_6, \{1,2\}) \circ K_n$;

(3) $\mathrm{Cay}(\mathbb{Z}_{iq}, \{1,i\}) \circ (\Sigma_x)_{x \in \mathbb{Z}_{iq}}$.

Here, $m \geq 1$, $n \geq 2$, $q \geq 4$, $i \in \{1,2\}$, $(\Sigma_x)_{x \in \mathbb{Z}_{iq}}$ are semicomplete weakly distance-regular digraphs with $p^{\tilde{h}}_{\tilde{i},\tilde{j}}(\Sigma_0) = p^{\tilde{h}}_{\tilde{i},\tilde{j}}(\Sigma_x)$ for each x and $\tilde{i}$, $\tilde{j}$, $\tilde{h}$, and Λ is a doubly regular $(k+1,2)$-team tournament of type II for a positive integer k with $k \equiv 3 \pmod 4$.

Y. Yang, S. Li and K. Wang, Locally semicomplete weakly distance-regular digraphs, arXiv: 2405.03310.

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P-polynomial case

A wdrdg is P-polynomial if its attached scheme is P-polynomial.

Theorem (Z., Yang and Wang, 2023)

Let Γ be a wdrdg whose attached scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ is P-polynomial with respect to the ordering $R_0, R_1, \ldots, R_d$. Then Γ is isomorphic to one of the following digraphs:

(i) (X, R_1) or (X, R_{g-1});

(ii) (X, R_2) or (X, R_{g-2}), $k_1 > k_g + 1$, $g \in \{6, 8\}$;

(iii) $(X, R_1 \cup R_2)$ or $(X, R_{g-2} \cup R_{g-1})$, $2 \mid g$;

(iv) $(X, R_1 \cup R_g)$ or $(X, R_{g-1} \cup R_g)$, $d = g$;

(v) $(X, R_2 \cup R_g)$ or $(X, R_{g-2} \cup R_g)$, $k_1 > k_g + 1$, $d = g$ and $g \in \{6, 8\}$;

(vi) $(X, R_1 \cup R_2 \cup R_g)$ or $(X, R_{g-2} \cup R_{g-1} \cup R_g)$, $d = g$, $2 \mid g$ and $g > 4$.

Here, k_i is the valency of the relation R_i for $i \in \{1, g\}$ and g is the girth of (X, R_1).

Q. Zeng, Y. Yang and K. Wang, P-polynomial weakly distance-regular digraphs, Electron. J. Combin., 30(3) (2023) Paper 3.3.

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Underlying graph is a DRG

Wang and Suzuki proposed a question when an orientation of a distance-regular graph defines a weakly distance-regular digraph.

K. Wang and H. Suzuki, Weakly distance-regular digraphs, Discrete Math. 264 (2003) 225–236.

For a digraph Γ, we form the underlying graph of Γ with the same vertex set, and there is an edge between vertices x, y whenever (x, y) or $(y, x) \in A(\Gamma)$.

Underlying graph is a complete graph

- A digraph is semicomplete if its underlying graph is a complete graph.
- A semicomplete wdrdg has diameter 2 and girth $g \leq 3$.
- 2-class non-symmetric AS $\Longleftrightarrow$ semicomplete wdrdg of girth 3.
- 3-class non-symmetric AS $\Longleftrightarrow$ semicomplete wdrdg of girth 2.

Hamming graph and Cartesian product

- Hamming graph $H(d,q)$
  Vertex set $X := F^d$, where $|F| = q$.
  $x \sim y$ iff x and y differ in 1 position.
  Hamming graphs also can be viewed as Cartesian products of complete graphs.
- For the digraphs Γ and Σ, the Cartesian product $\Gamma \square \Sigma$ is the digraph with the vertex set $V(\Gamma) \times V(\Sigma)$ such that $((u,v),(u',v'))$ is an arc if and only if $u = u'$ and $(v,v') \in A(\Sigma)$, or $(u,u') \in A(\Gamma)$ and $v = v'$.

Folded n-cube, Shrikhande graph and Doob graph

- Folded n-cube $\square_n$
  The graph $H(n-1,2)$ with a perfect matching introduced between antipodal vertices, where two vertices are called the antipodal vertices if they differ in all coordinates.
- Shrikhande graph
  $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{(\pm 1, 0), (0, \pm 1), \pm(1,1)\})$.
- Doob graph $G(d_1, d_2)$
  The Cartesian product of $H(d_2, 4)$ with d_1 copies of the Shrikhande graph.

Induced subdigraph

- Γ is a commutative wdrdg with vertex set S^d.
- For each $i \in \{1, 2, \ldots, d\}$ and $a_j \in S$ with $1 \leq j \leq d-1$, denote $\Gamma_i(a_1, a_2, \ldots, a_{d-1})$ be the induced subdigraph of Γ on the set

$$\{(a_1, a_2, \ldots, a_{i-1}, b, a_i, a_{i+1}, \ldots, a_{d-1}) \mid b \in S\}.$$

- An arc (x, y) of Γ is of type $(1, r)$ if $\partial(y, x) = r$.

Proposition (Yang, Z. and Wang, 2024)

Let $q = 2$. If Σ is distance-transitive, then $p^{(1,3)}_{(2,2),(3,1)} \neq 0$ and each arc of Γ is of type $(1, 1)$ or $(1, 3)$. In particular, $k_{1,1} = 0$ or 1.

Y. Yang, Q. Zeng and K. Wang, Weakly distance-regular digraphs whose underlying graphs are distance-regular, I, J. Algebraic Combin., (2024). https://doi.org/10.1007/s10801-024-01312-3

Underlying graph is a Hamming graph

Theorem (Yang, Z. and Wang, 2024)

Let Γ be a commutative wdrdg. Then Γ has a Hamming graph $H(d,q)$ as its underlying graph if and only if Γ is isomorphic to one of the following digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_4, \{1\})$;

(2) $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0),(0,1)\})$;

(3) Δ^1;

(4) $\Delta^1 \square \Delta^2$;

(5) $\Gamma^1 \square \Gamma^2 \square \cdots \square \Gamma^d$.

Here, $d \geq 1$, and $(\Delta^i)_{i\in\{1,2\}}$ (resp. $(\Gamma^i)_{i\in\{1,2,\ldots,d\}}$) are semicomplete weakly distance-regular digraphs of diameter 2 and girth 2 (resp. 3) with $p^{\tilde{h}}_{\tilde{i},\tilde{j}}(\Delta^1) = p^{\tilde{h}}_{\tilde{i},\tilde{j}}(\Delta^2)$ (resp. $p^{\tilde{h}}_{\tilde{i},\tilde{j}}(\Gamma^1) = p^{\tilde{h}}_{\tilde{i},\tilde{j}}(\Gamma^l)$) for each l and $\tilde{h}, \tilde{i}, \tilde{j}$.

Y. Yang, Q. Zeng and K. Wang, Weakly distance-regular digraphs whose underlying graphs are distance-regular, I, J. Algebraic Combin., (2024).

Underlying graph is a Doob graph or folded n-cube

Theorem (Yang, Z. and Wang, 2024)

Let Γ be a commutative wdrdg. Then Γ has a folded n-cube as its underlying graph if and only if Γ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_4, \{1, 2\})$.

Theorem (Yang, Z. and Wang, 2024)

Let Γ be a commutative wdrdg. Then Γ has a Doob graph $G(d_1, d_2)$ as its underlying graph if and only if Γ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{(1, 0), (0, 1), (-1, -1)\})$.

Y. Yang, Q. Zeng and K. Wang, Weakly distance-regular digraphs whose underlying graphs are distance-regular, I, J. Algebraic Combin., (2024). https://doi.org/10.1007/s10801-024-01312-3

Outline

1. Introduction
   - Digraph
   - Association scheme
   - Distance-regular graphs
   - Distance-regular digraph
   - Weakly distance-regular digraphs
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3. Our works
   - P-polynomial case
   - Underlying graphs are Hamming graphs or related graphs
   - Underlying graphs are Johnson graphs or related graphs

Johnson graph and folded Johnson graph

- Johnson graph $J(n,e)$
  Vertex set X: the set of all e-subsets of an n-set.
  $x \sim y$ iff $|x \cap y| = e-1$.
- folded Johnson graph $\bar{J}(2m,m)$
  Vertex set X: the set of partitions of a $2m$-set into two m-sets.
  Two partitions being adjacent whenever their common refinement is a partition of X into four sets of sizes 1, $m-1$, 1, $m-1$.

The type of arcs

- Γ is a commutative wdrdg whose underlying graph is Σ, where Σ is a Johnson graph $J(n,e)$ ($n \geq 2e$ and $e \geq 2$) or a folded Johnson graph $\bar{J}(2m,m)$ ($m \geq 4$).
- $T = \{q \mid (1,q-1) \in \tilde{\partial}(\Gamma)\}$.

Proposition (Z., Yang and Wang, 2024^+)

Let $q \in T$. If $p^{(1,q-2)}_{(1,q-1),(1,h-1)} = 0$ for all $h \geq 2$, then $q \leq 4$.

Q. Zeng, Y. Yang and K. Wang, Weakly distance-regular digraphs whose underlying graphs are distance-regular, II, in preparation.

$|T| > 1$ or $|T| = 1$

Proposition (Z., Yang and Wang, 2024^+)

Suppose $|T| > 1$. Then $\Sigma = J(4,2)$ and Γ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_6, \{1,2\})$.

Proposition (Z., Yang and Wang, 2024^+)

Suppose $|T| = 1$. Then $\Sigma = J(4,2)$ and Γ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_6, \{1,4\})$.

Q. Zeng, Y. Yang and K. Wang, Weakly distance-regular digraphs whose underlying graphs are distance-regular, II, in preparation.

Underlying graph is a Johnson graph or folded Johnson graph

Theorem (Z., Yang and Wang, 2024^+)

Let Γ be a commutative wdrdg and Σ a Johnson graph $J(n, e)$ with $n \geq 2e$ and $e \geq 2$. Then Γ has Σ as its underlying graph if and only if Γ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_6, \{1, 2\})$ or $\mathrm{Cay}(\mathbb{Z}_6, \{1, 4\})$.

Theorem (Z., Yang and Wang, 2024^+)

Let Γ be a commutative wdrdg and Σ a folded Johnson graph $\bar{J}(2m, m)$. Then Γ dose not have Σ as its underlying graph if $m \geq 4$.

Q. Zeng, Y. Yang and K. Wang, Weakly distance-regular digraphs whose underlying graphs are distance-regular, II, in preparation.

Further work

- Classify commutative wdrdgs whose underlying graphs are some other distance-regular graphs.
- Classify wdrdgs whose underlying graphs are Hamming graphs or Johnson graphs.
- Consider wdrdgs whose attached association schemes are Q-polynomial.
- Classify weakly distance-regular dihedrants.

Thank you for your attention.