Graph growth

of permutation groups

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MARSDEN FUND
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ROYAL SOCIETY TE APĀRANGI

Seminar on Groups and Graphs

8th May 2024

UNIVERSITY OF AUCKLAND
Waipapa Taumata Rau
NEW ZEALAND

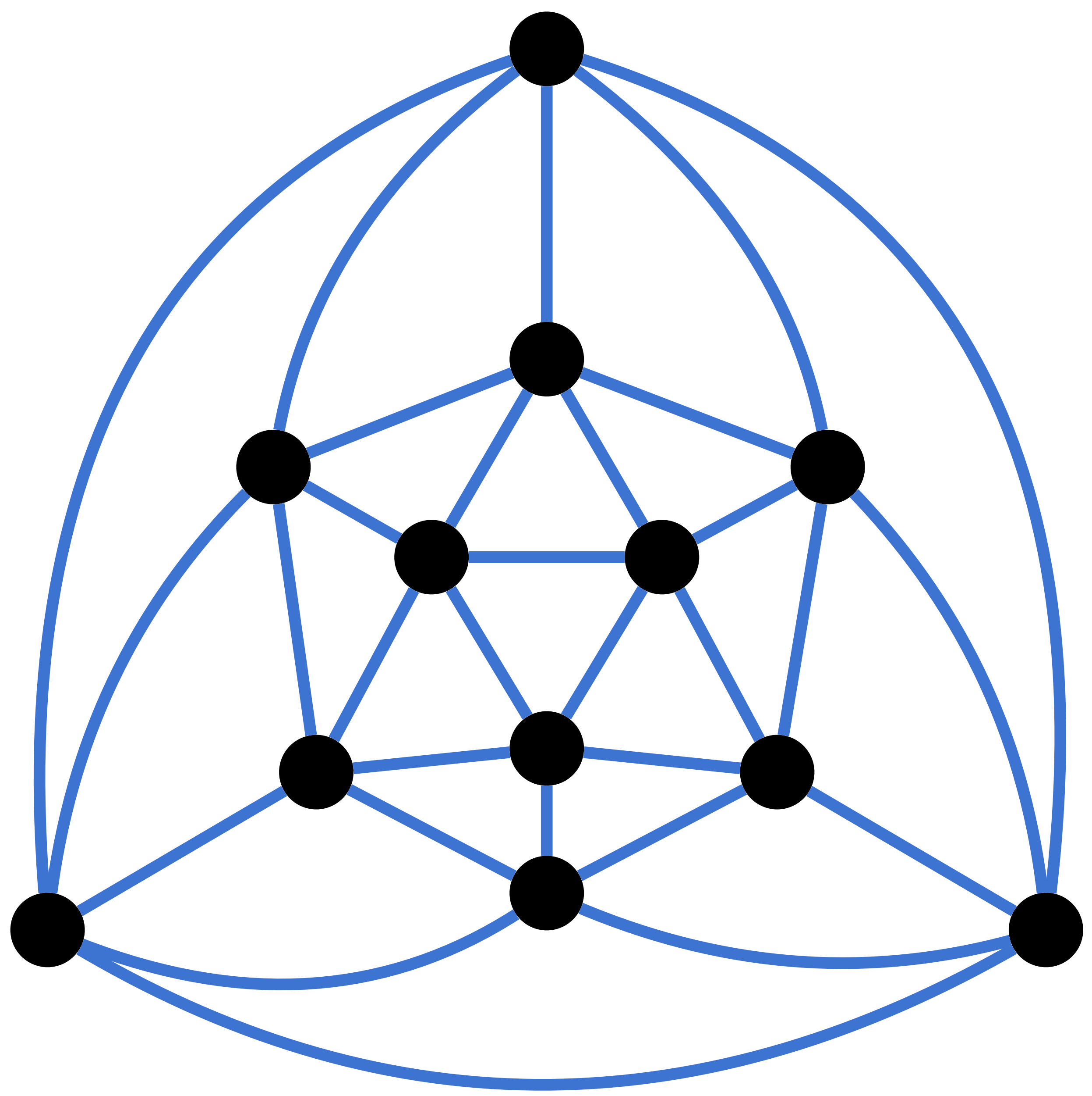
Graphs are **finite, simple** (undirected, no multiple edges, no loops) and **connected**

Automorphisms of a graph $\Gamma = (V, E)$ are permutations of V preserving E

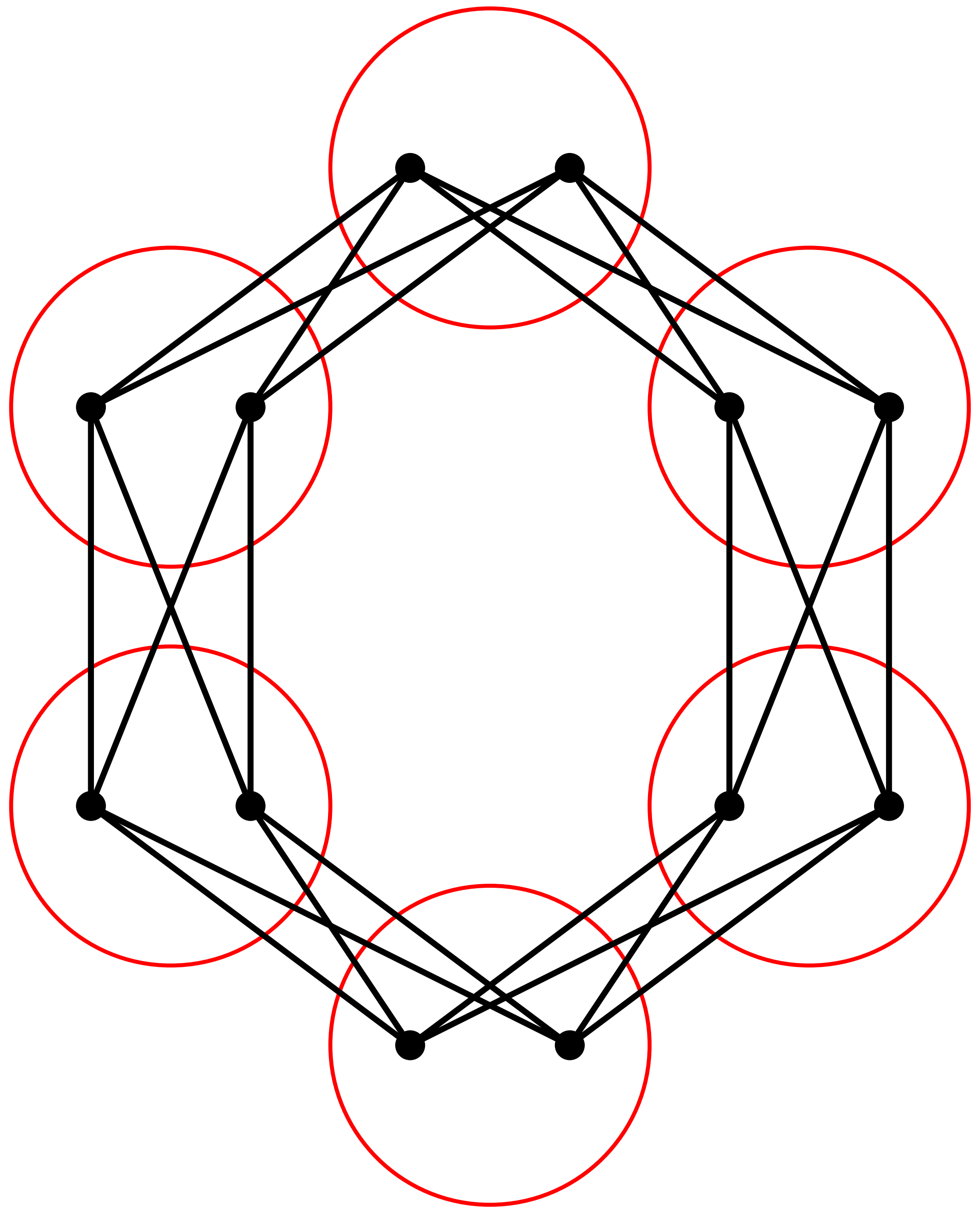
A graph Γ is **arc-transitive** if $\mathrm{Aut}(\Gamma)$ is transitive on arcs (directed edges)

$$\forall (x, y), (z, w) \text{ with } \{x, y\}, \{z, w\} \in E(\Gamma)$$

$$\exists g \in \mathrm{Aut}(\Gamma) \text{ s.t. } (x, y)^g = (z, w)$$



Valency is the number of neighbours of a vertex



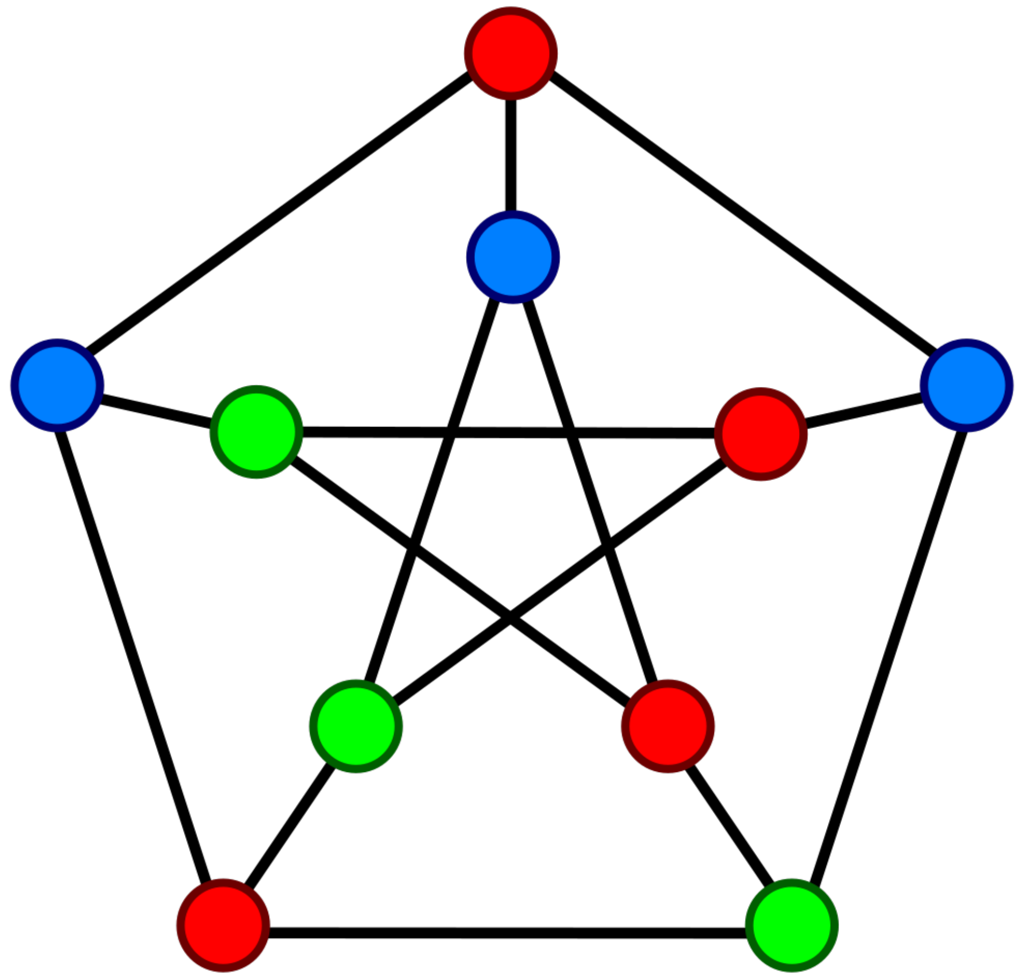
Tutte's theorem

Let Γ be a finite 3-valent, connected graph and

$G \leq \mathrm{Aut}(\Gamma)$ transitive on the arcs of Γ.

$|G_v| \leq 48.$

$P =$



$$\mathrm{Aut}(P) \cong_{perm} S_5 \curvearrowright \binom{\{1,\ldots,5\}}{2}$$

$$\mathrm{Aut}(P)_v \cong S_3 \times S_2$$

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⬇

Census of 3-valent arc-transitive graphs on ≤ 10000 vertices (Conder)

Asymptotic enumeration (Potočnik, Spiga, Verret)

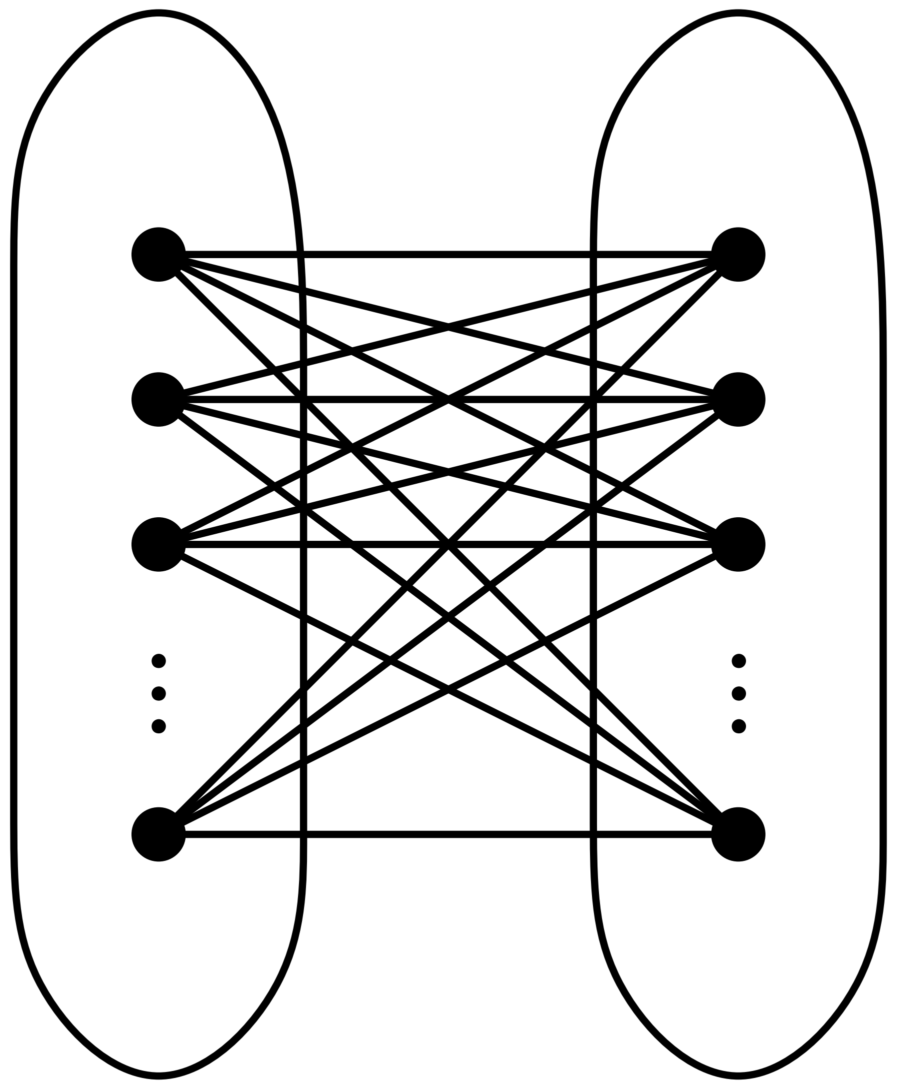
...

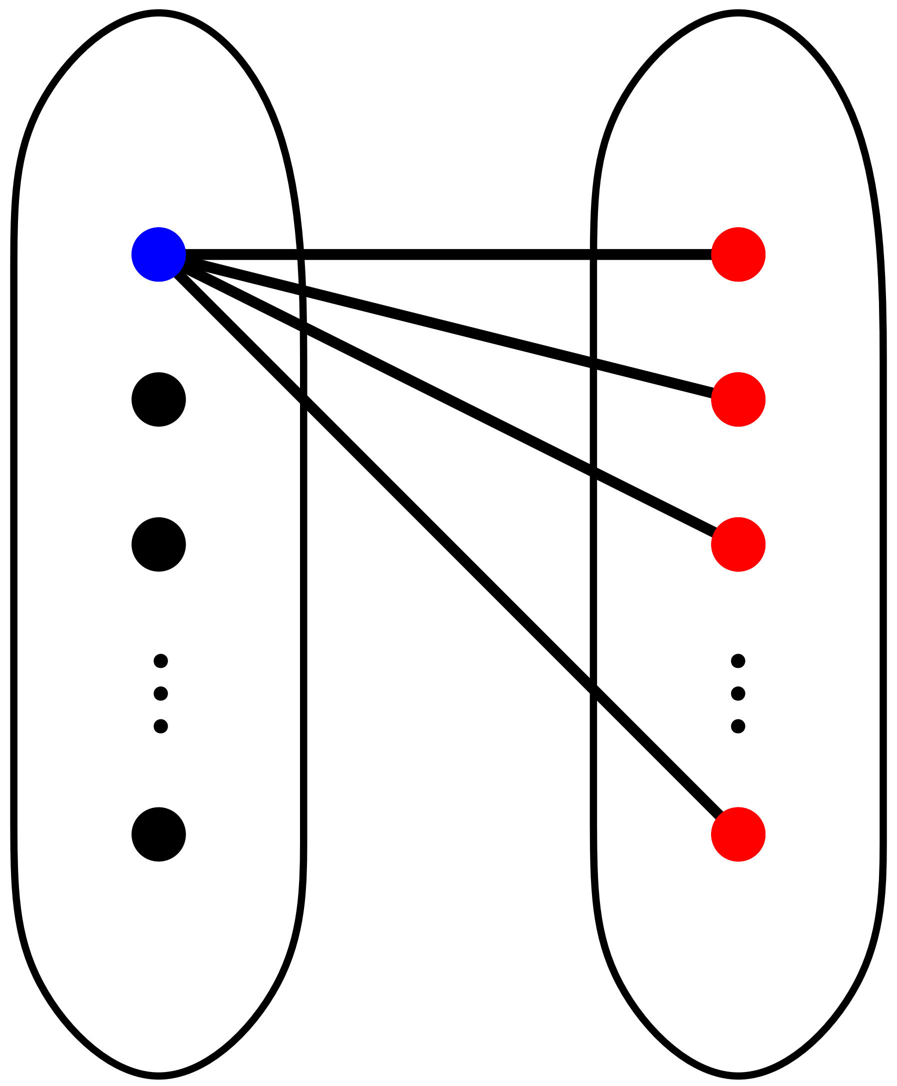
Local action

Γ is a connected graph, $G \leq \mathrm{Aut}(\Gamma)$ vertex-transitive.

The pair (Γ, G) is locally-L if $G_v \curvearrowright \Gamma(v)$ induces L.

If $\Gamma = K_{n,n}$ and $G = \mathrm{Aut}(K_{n,n}) \cong S_n \wr S_2$, then $G_v = S_{n-1} \times S_n$ and $L = S_n$.





The pair $(K_{n,n}, S_n \wr S_2)$ is locally-S_n.

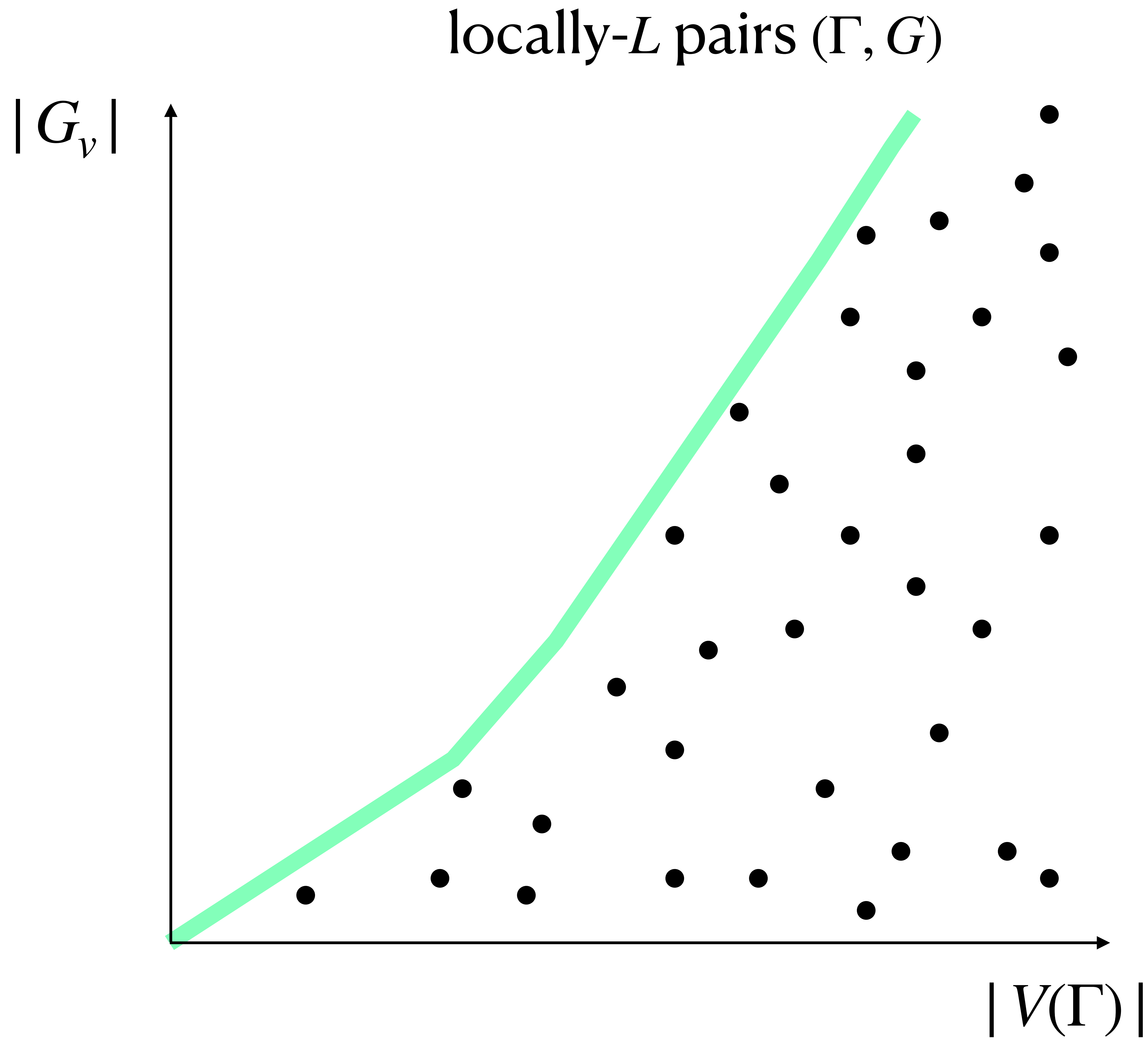
Graph growth

L = permutation group

Graph growth of L

is the growth rate of $|G_v|$ wrt $|V(\Gamma)|$

in locally-L pairs (Γ, G).



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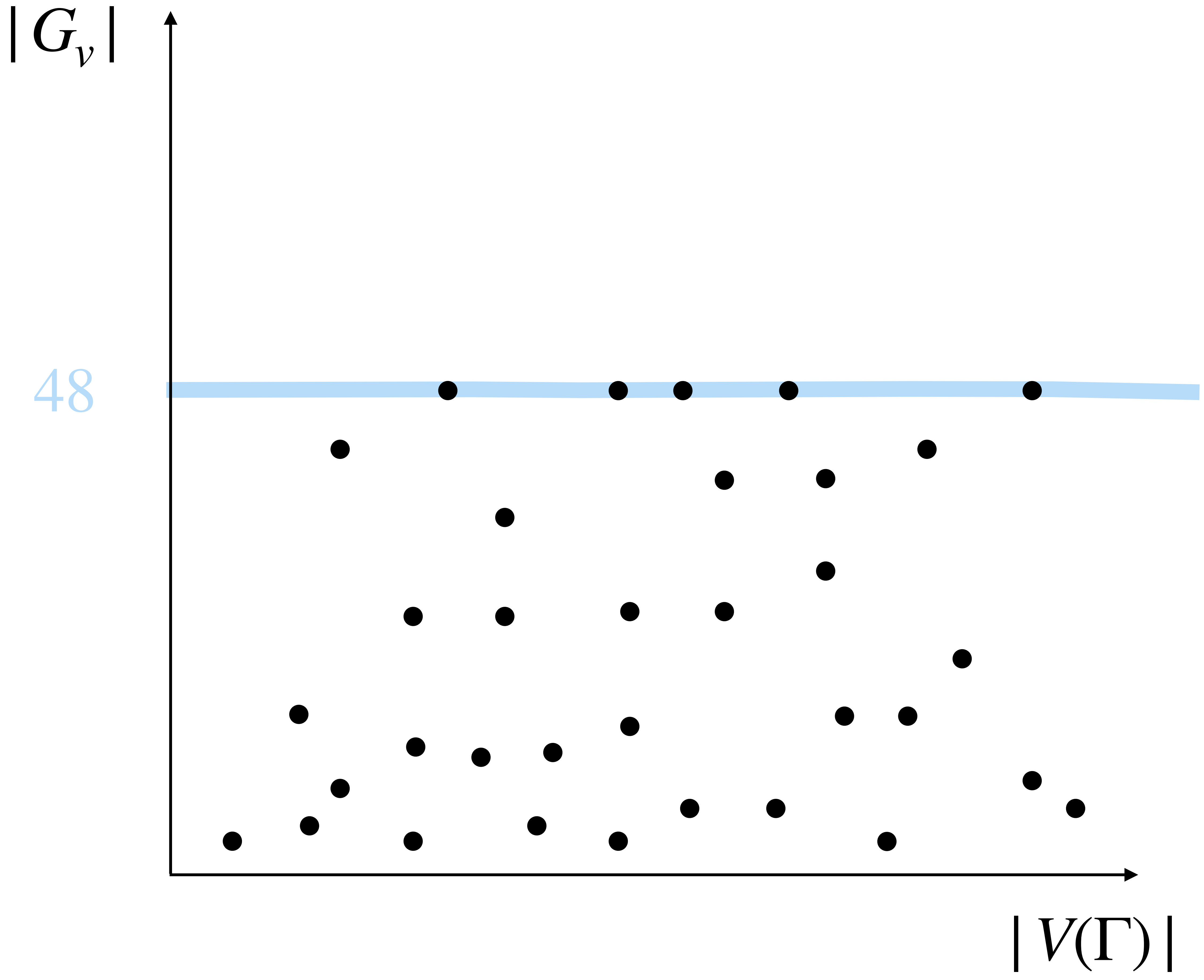
Tutte's theorem

A_3 and S_3 have Constant growth

locally-A_3 and locally-S_3 pairs (Γ, G)



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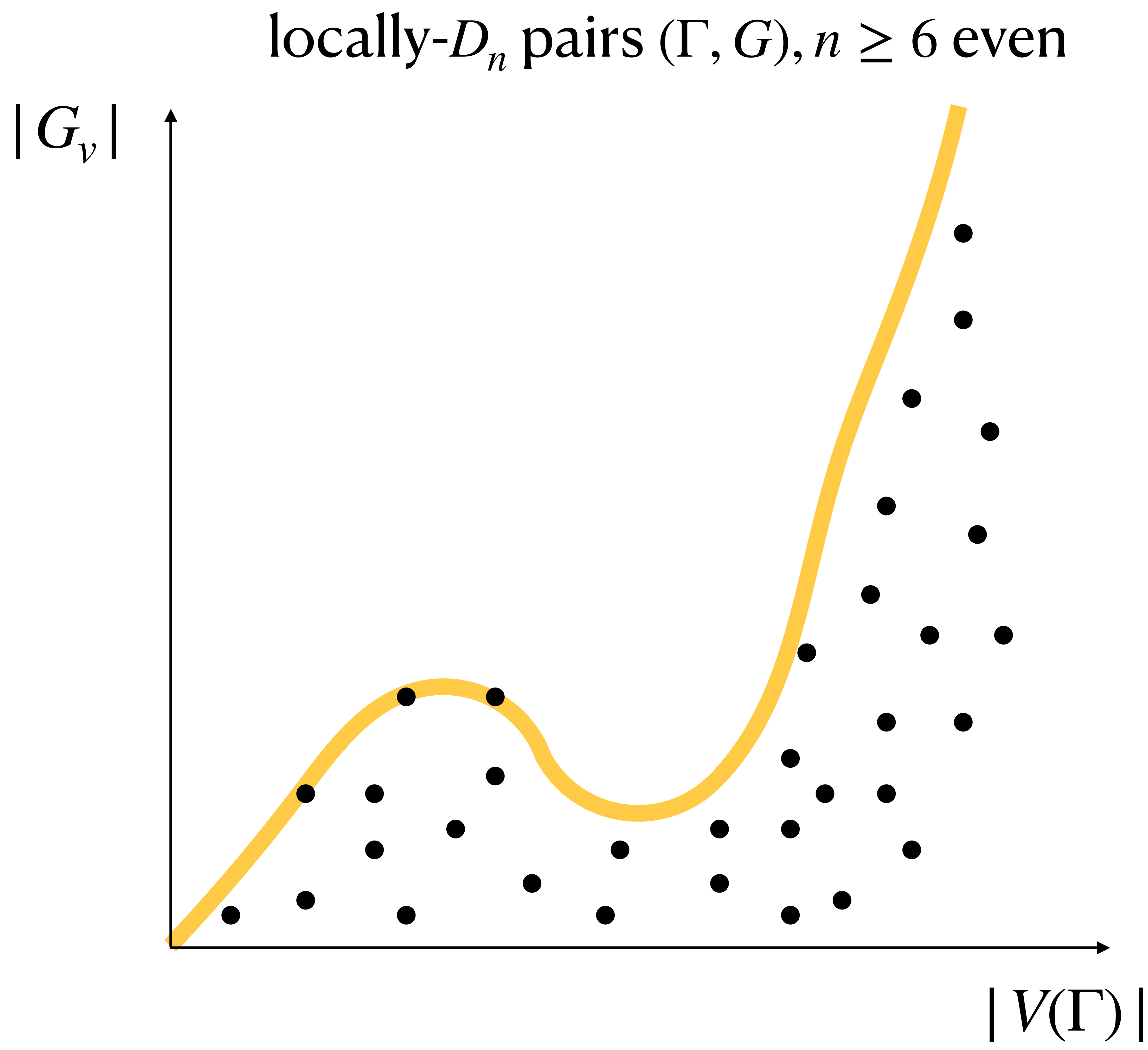
For even $n \geq 6$, D_n has Polynomial growth



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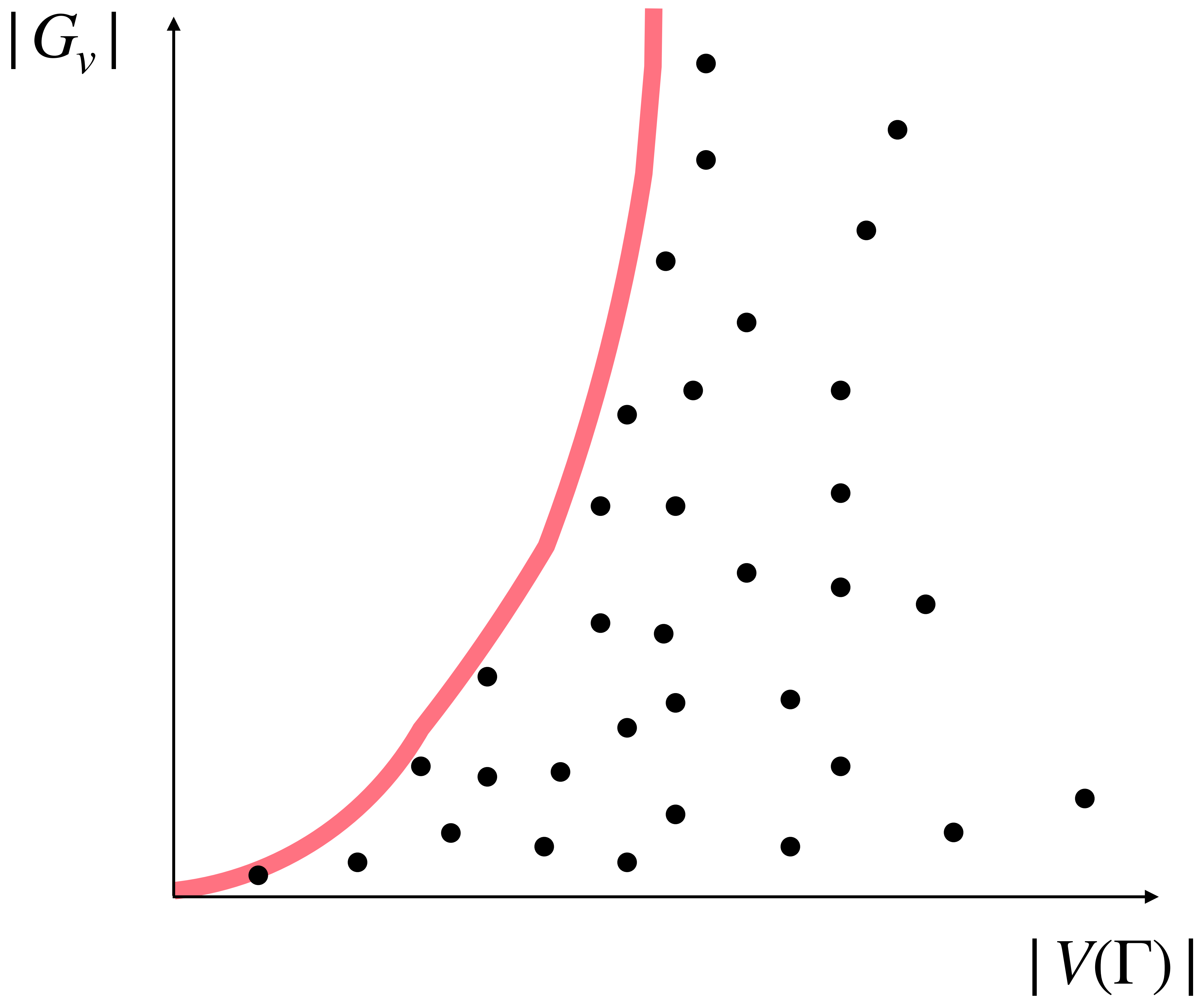
Imprimitive wreath products

have Exponential growth

locally-L pairs (Γ, G) with L an imp. wr. prod.



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Imprimitive wreath products

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Remarks

"Slow" graph growth still allows

for cool results.

Graph growth is always at most Exponential.

This might not be the full list

of types of graph growth!

My PhD project

Graph growth of transitive permutation groups

Degree ≤ 7

Degree	Constant	Polynomial	Exponential
2	S_2	/	/
3	A_3, S_3	/	/
4	C_4, V_4, A_4, S_4	/	D_4
5	C_5, D_5 F_5, A_5, S_5	/	/
6	six groups	D_6	nine groups
7	seven groups	/	/

Graph growth of transitive permutation groups of degree 8

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Graph growth	#	Groups
Constant	14	semiprimitive groups
Polynomial	3	D_8 SD_8 $S_4 \curvearrowright cube$
Exponential	30	various imprimitive groups

Every normal subgroup is transitive or semiregular.

Graph growth of transitive permutation groups of degree 8

Graph growth	#	Groups
Constant	14	semiprimitive groups
Polynomial	3	D_8 SD_8 $S_4 \curvearrowright cube$
Exponential	30	various imprimitive groups
UNKNOWN	3	$V_0(\mathbb{F}_2^4) \rtimes A_4$ $V_0(\mathbb{F}_2^4) \rtimes S_4$ $V_0(\mathbb{F}_2^4) \,.\, S_4$

Every normal subgroup is transitive or semiregular.

Three transitive groups of degree 8

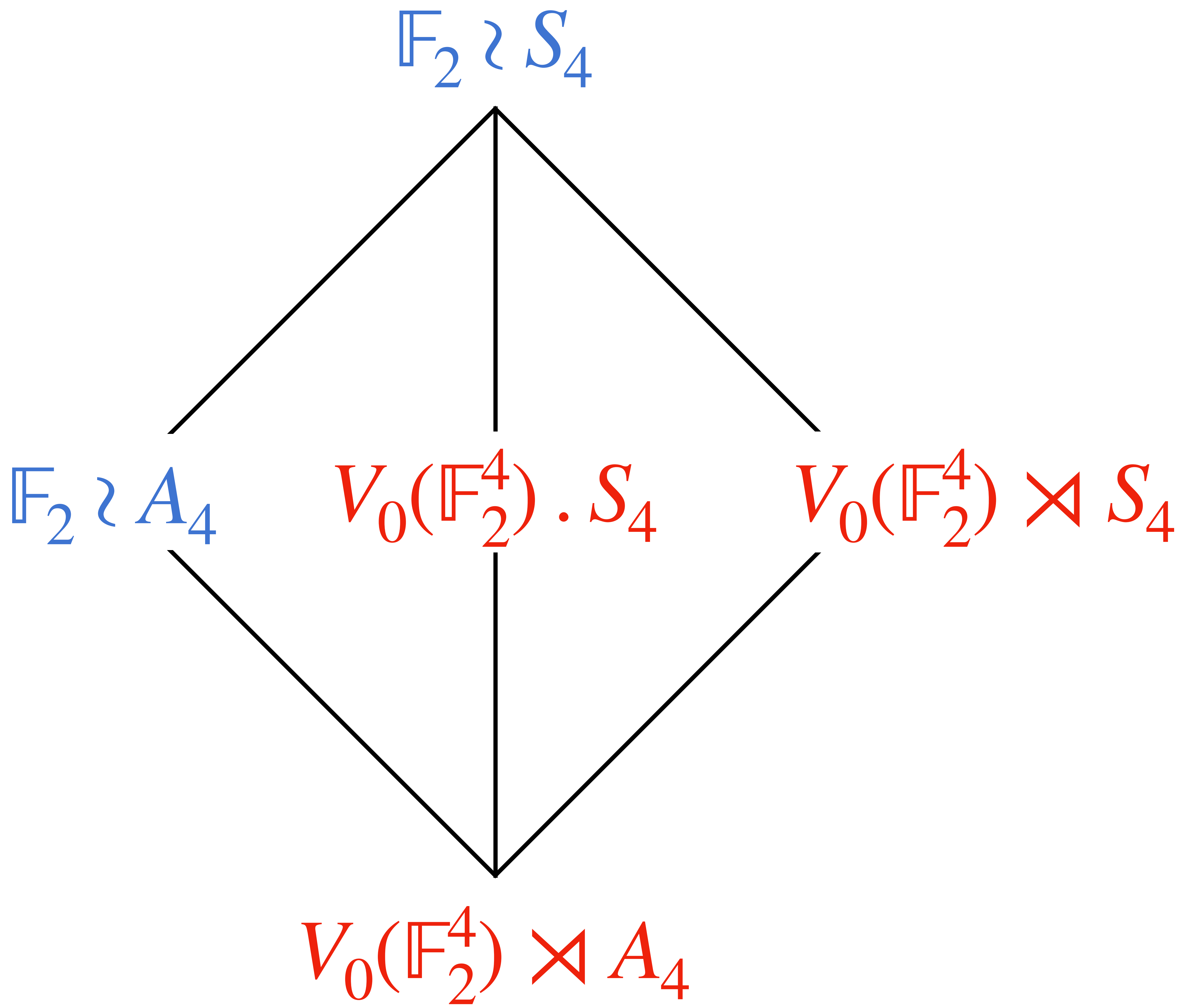
- Exponential graph growth
- Unknown graph growth

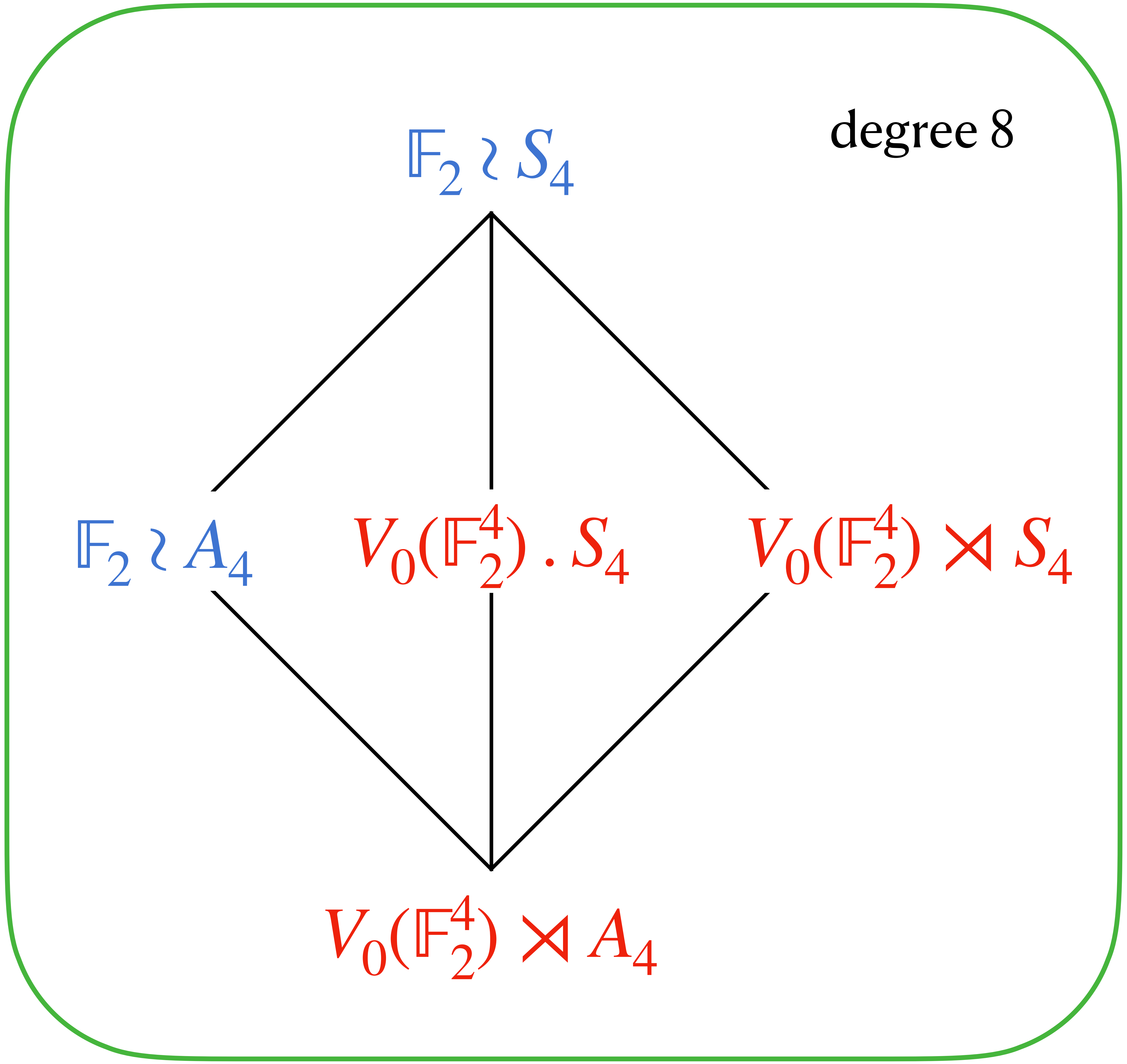
Admit a unique block system

with four blocks of size 2

Deleted permutation module

$$V_0(\mathbb{F}^n) = \{(v_1, \ldots, v_n) \in \mathbb{F}^n \mid \sum_{i=1}^{n} v_i = 0\}$$





degree 8
$\mathbb{F}_2 \wr S_4$
$\mathbb{F}_2 \wr A_4$
$V_0(\mathbb{F}_2^4) . S_4$
$V_0(\mathbb{F}_2^4) \rtimes S_4$
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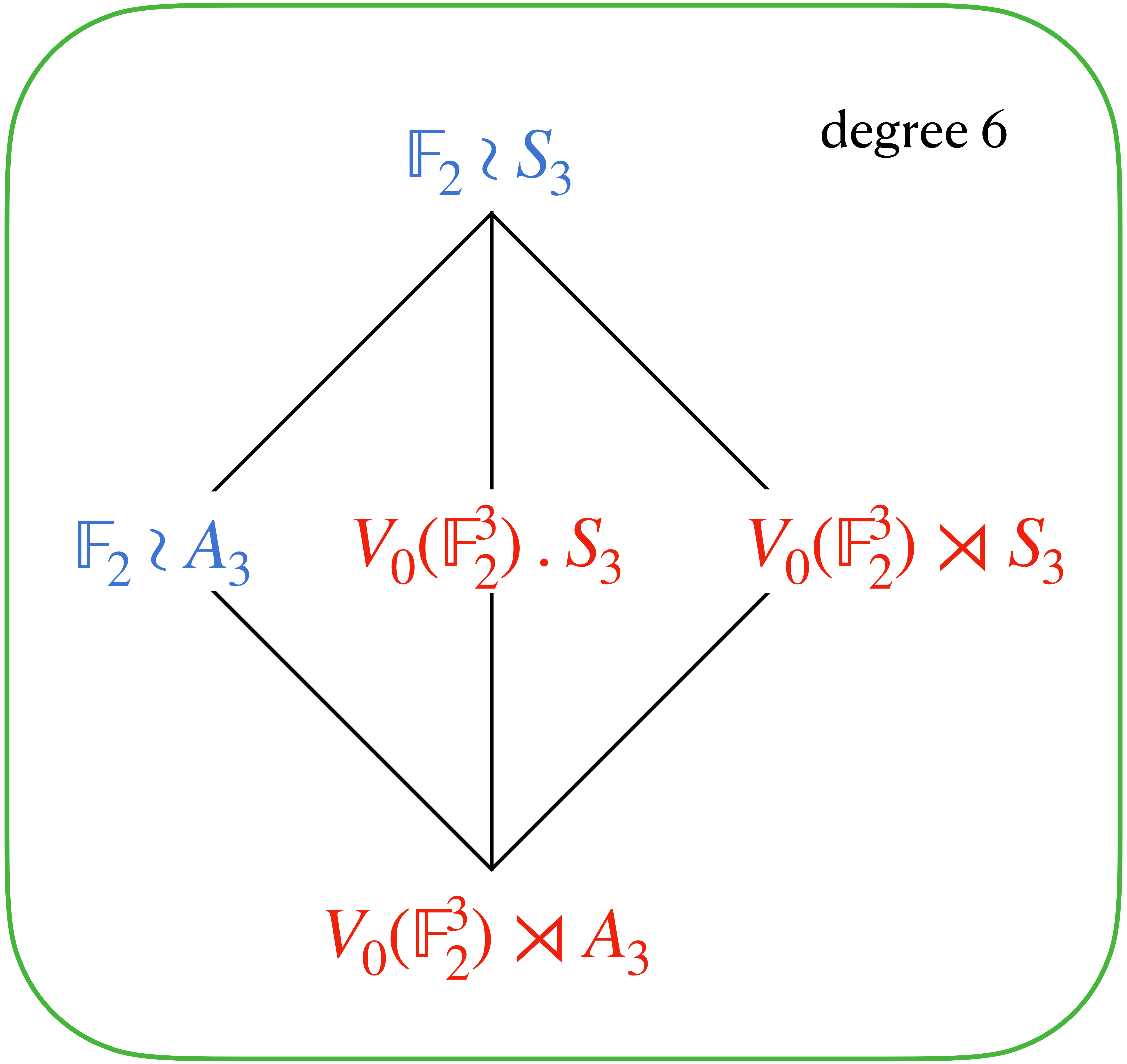
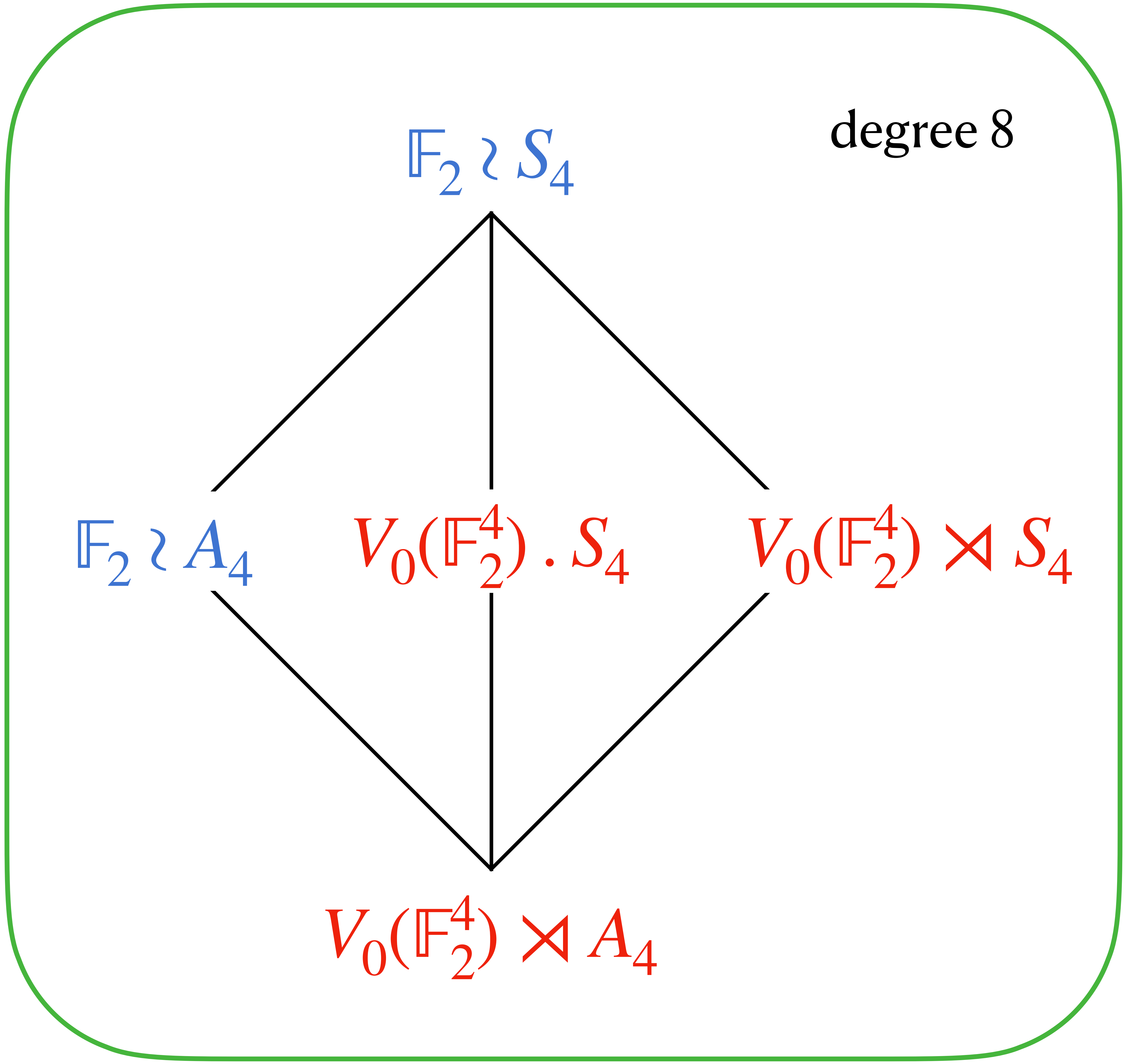
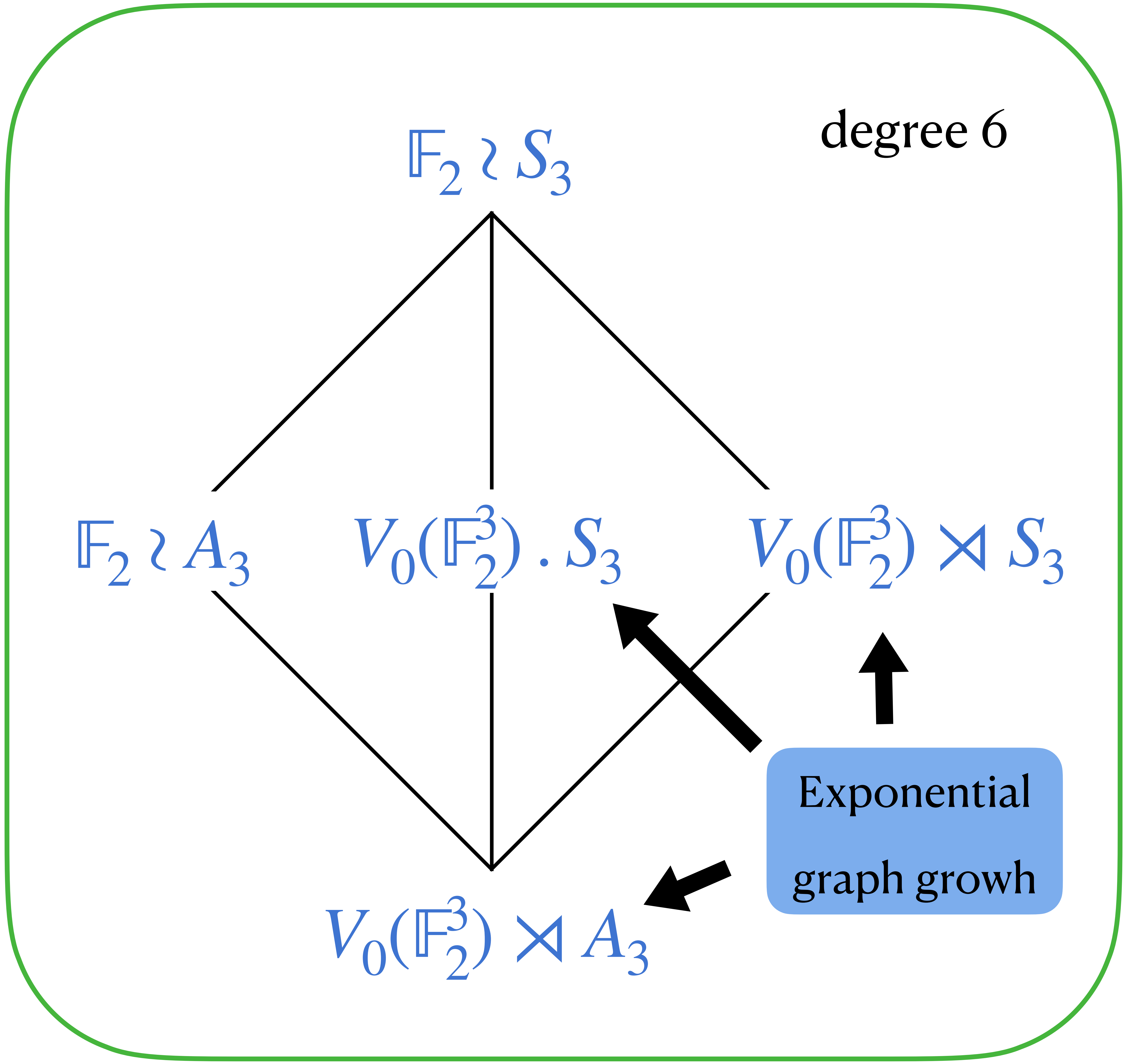


degree 6
$\mathbb{F}_2 \wr S_3$
$\mathbb{F}_2 \wr A_3$
$V_0(\mathbb{F}_2^3) . S_3$
$V_0(\mathbb{F}_2^3) \rtimes S_3$
$V_0(\mathbb{F}_2^3) \rtimes A_3$

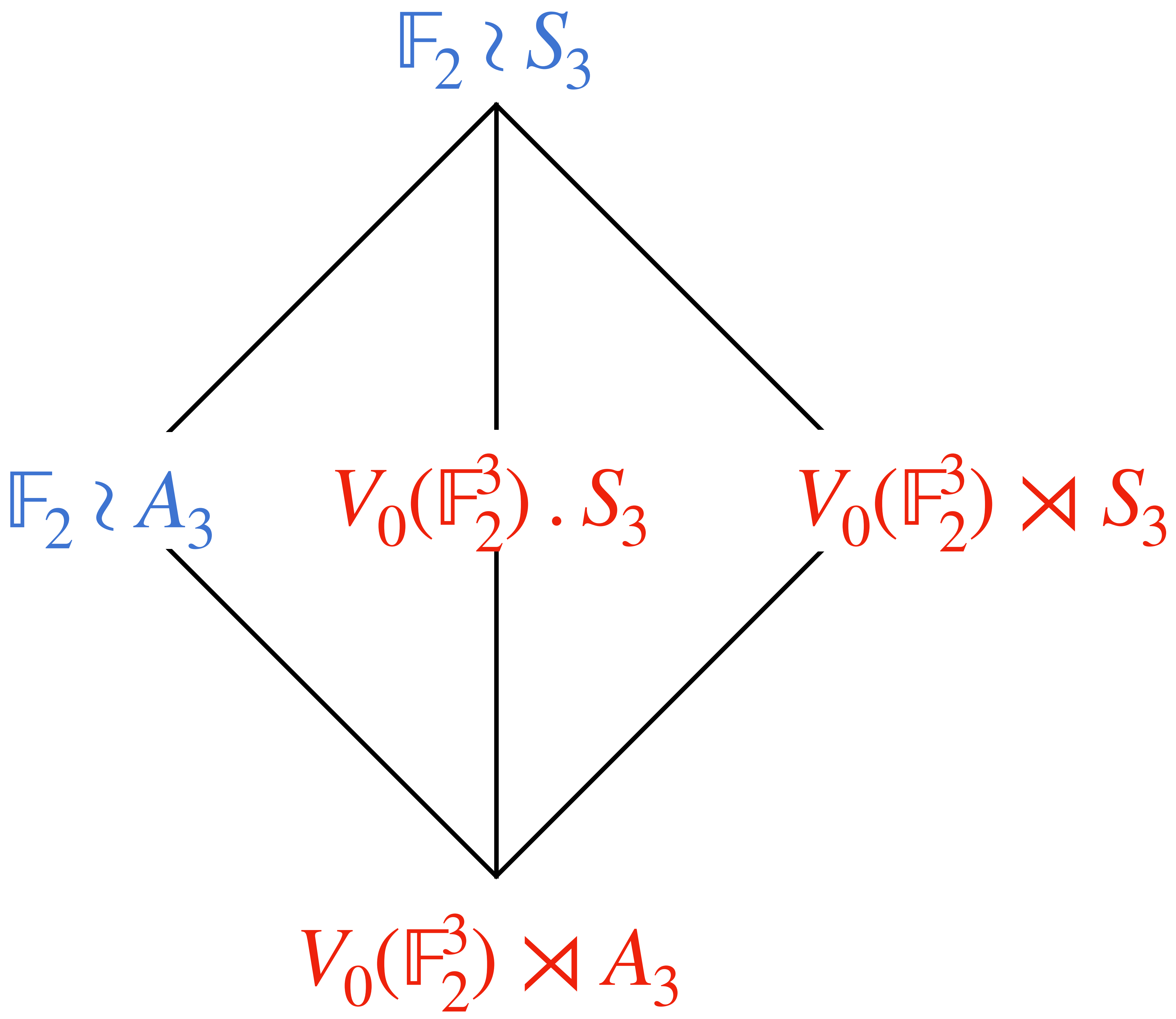


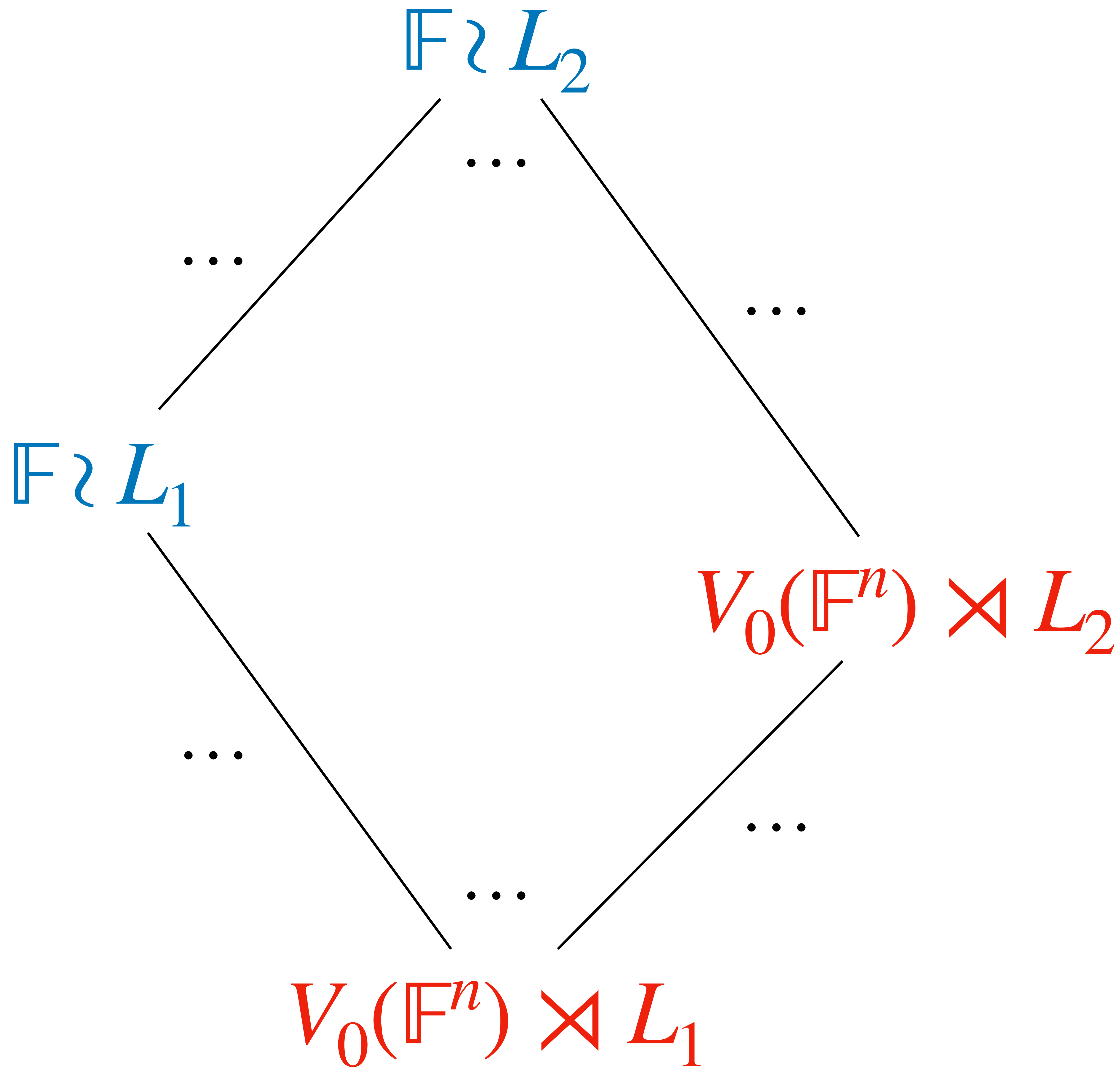


Hujdurović, Potočnik, Verret (2022)

$L_1 \leq L_2$ = transitive permutation groups of degree n

$\mathbb{F}$ = finite field of order q





$n = 3, L_1 = A_3, L_2 = S_3$ and $\mathbb{F} = \mathbb{F}_2$

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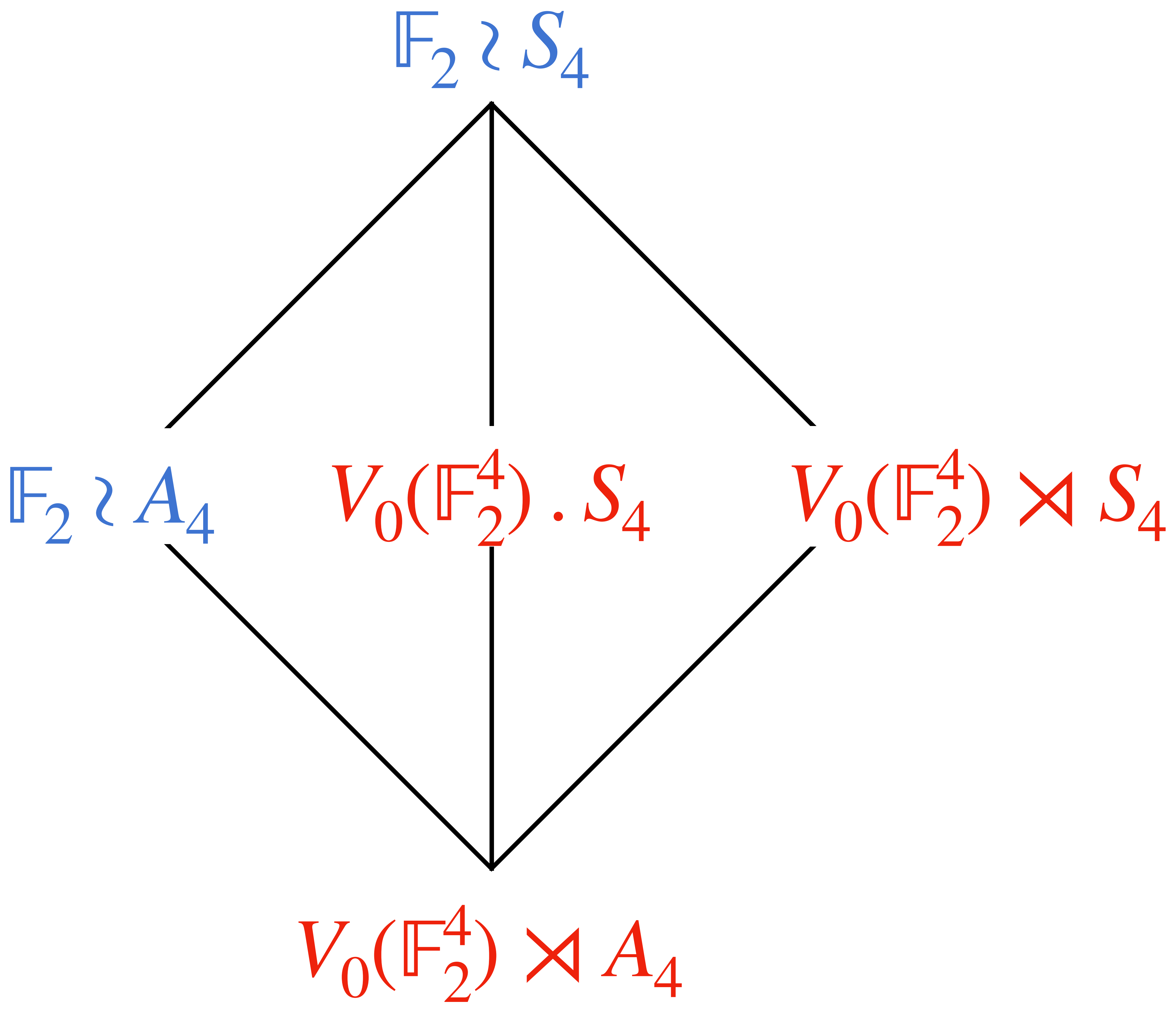
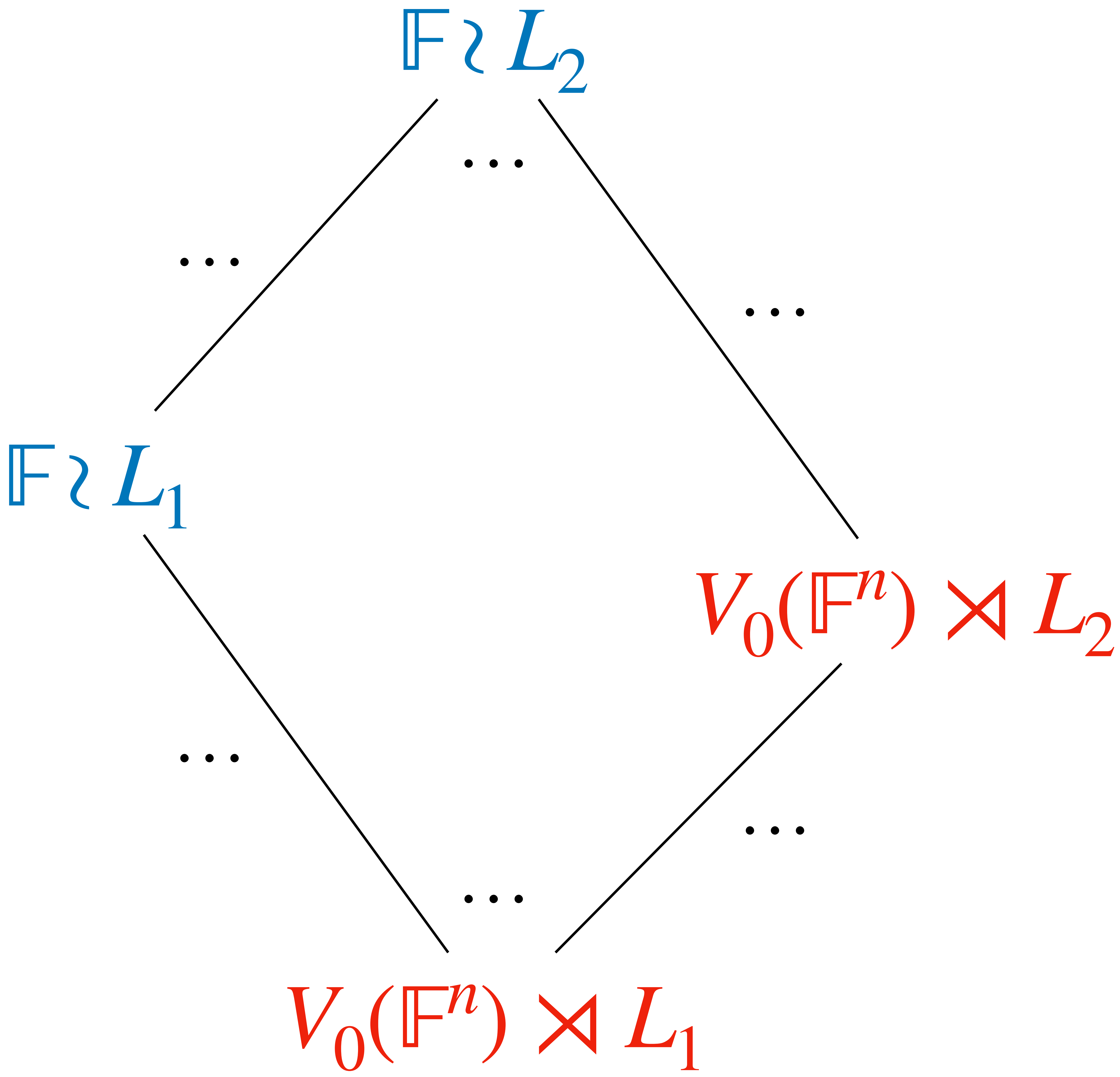




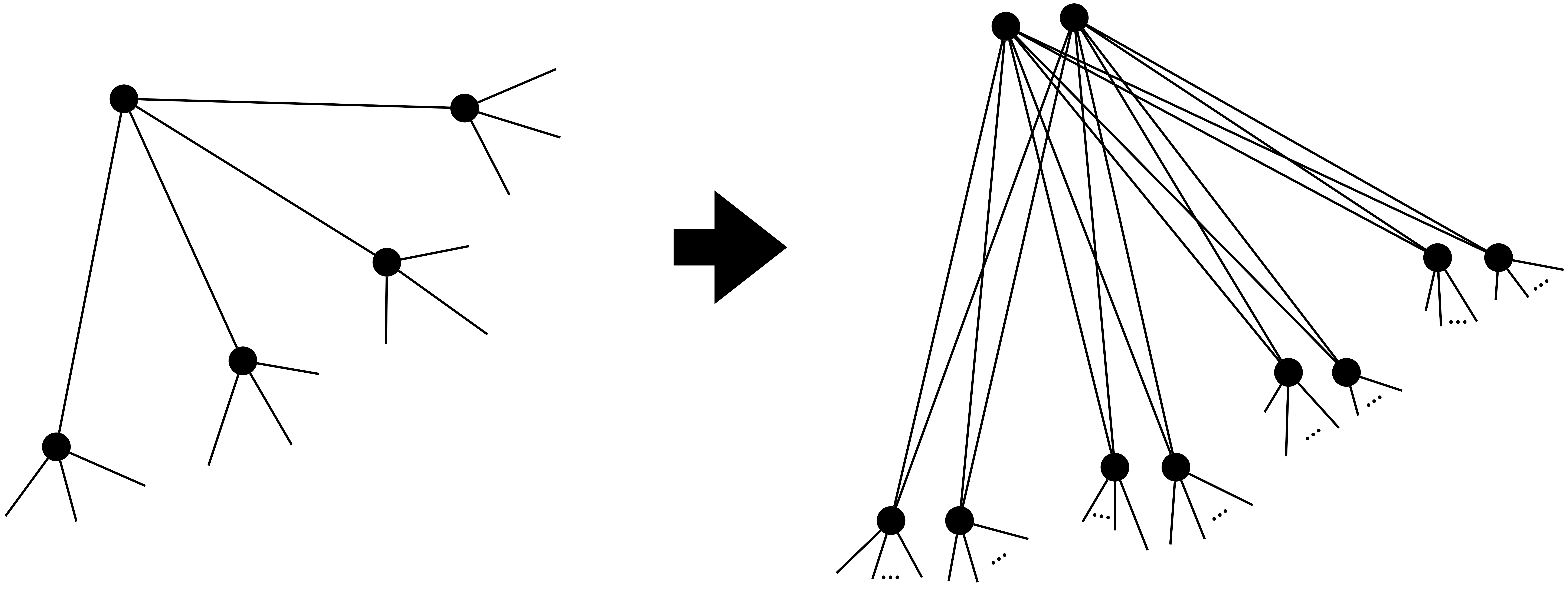
$n = 4, L_1 = A_4, L_2 = S_4$ and $\mathbb{F} = \mathbb{F}_2$

Locally-$(V_0(\mathbb{F}^n) \rtimes L_1)$ pairs

via lexicographic products

Take $n = 4, \mathbb{F} = \mathbb{F}_2$ and $L_1 = A_4$.

$L = V_0(\mathbb{F}_2^4) \rtimes A_4$ has a block system with four blocks of size two.



X $\quad$ $Y = X[\overline{K_2}]$

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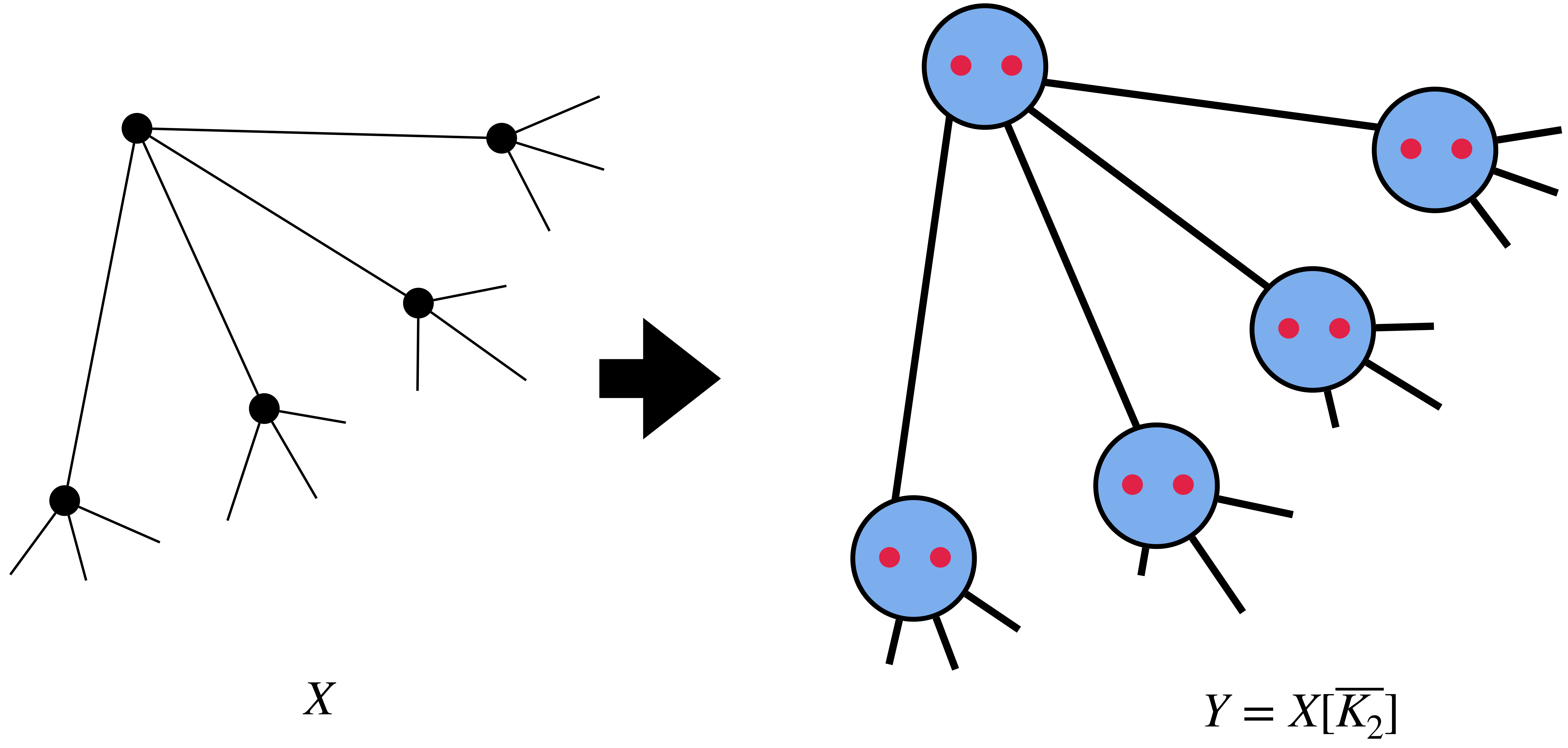

$\mathbb{F}_2 \wr \text{Aut}(X) = \mathbb{F}_2^{|V(X)|} \rtimes \text{Aut}(X)$
$\leq \text{Aut}(Y)$

Can we find an
$C \leq \mathbb{F}_2 \wr \text{Aut}(X)$ s.t.
(Y, C) is locally-L?

$C = M \rtimes H$

1. $H \leq \text{Aut}(X)$ locally inducing A_4
2. $M \leq \mathbb{F}_2^{|V(X)|}$ an H-module locally inducing $V_0(\mathbb{F}_2^4)$

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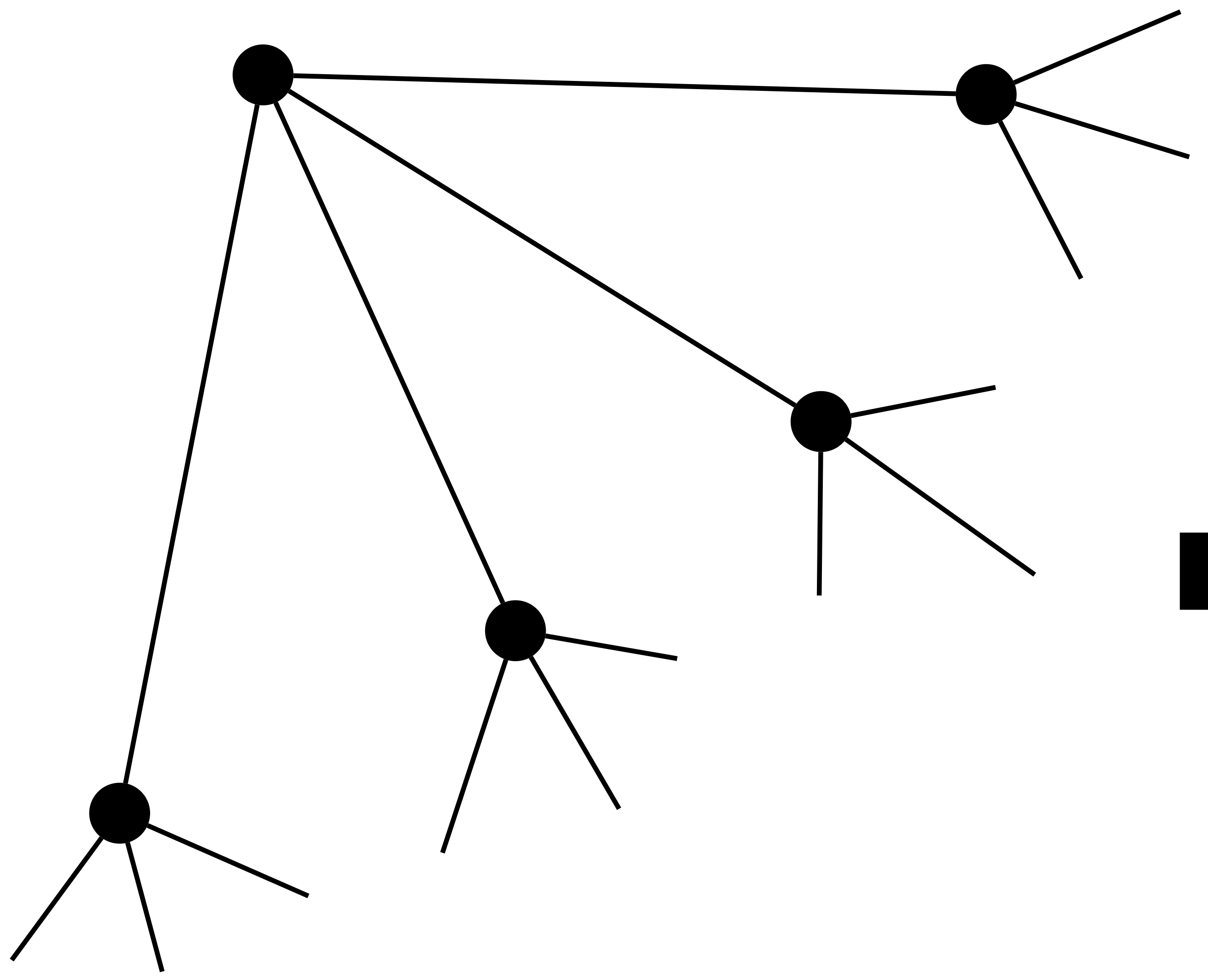
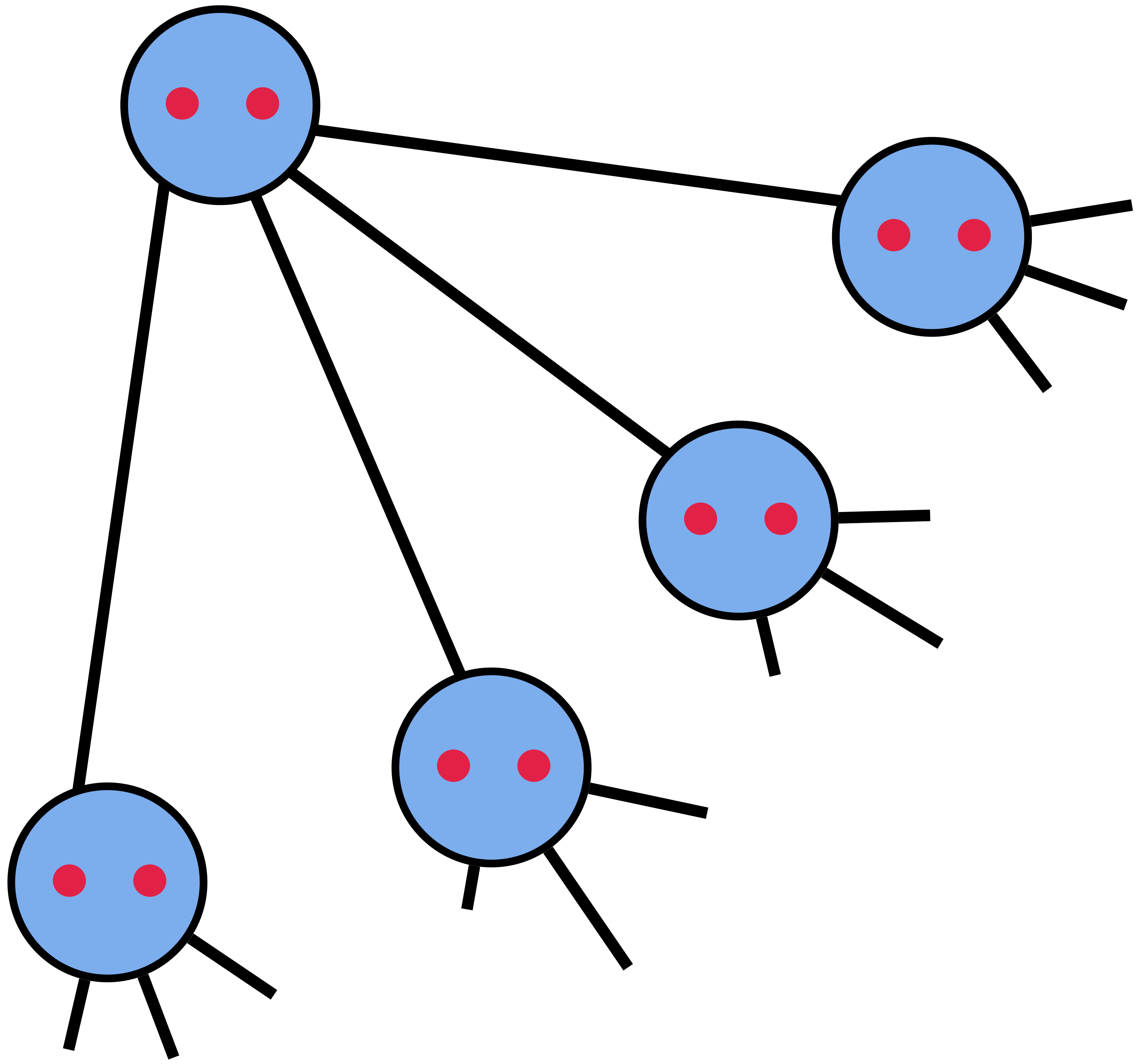


(X, H) is locally-A_4



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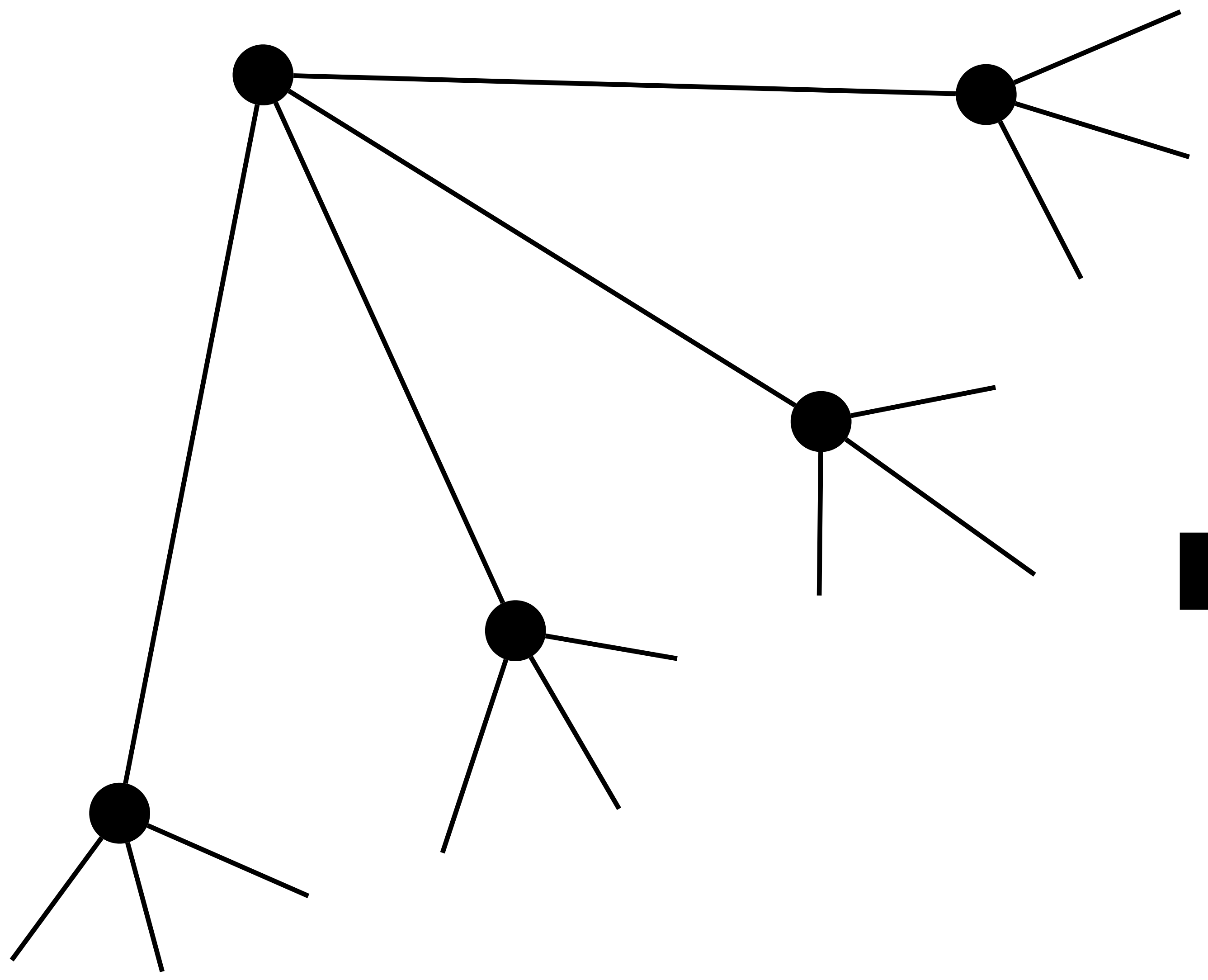

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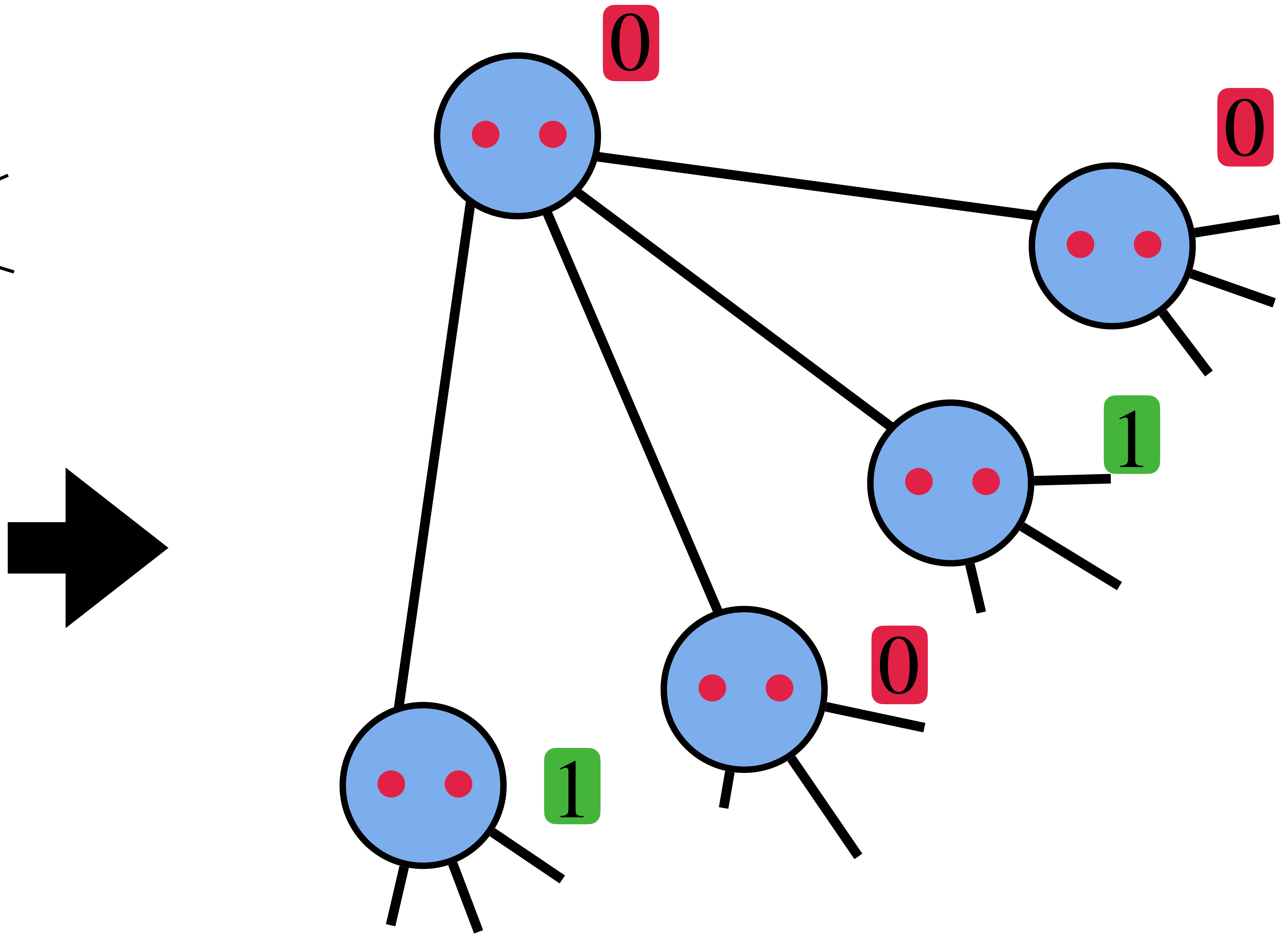

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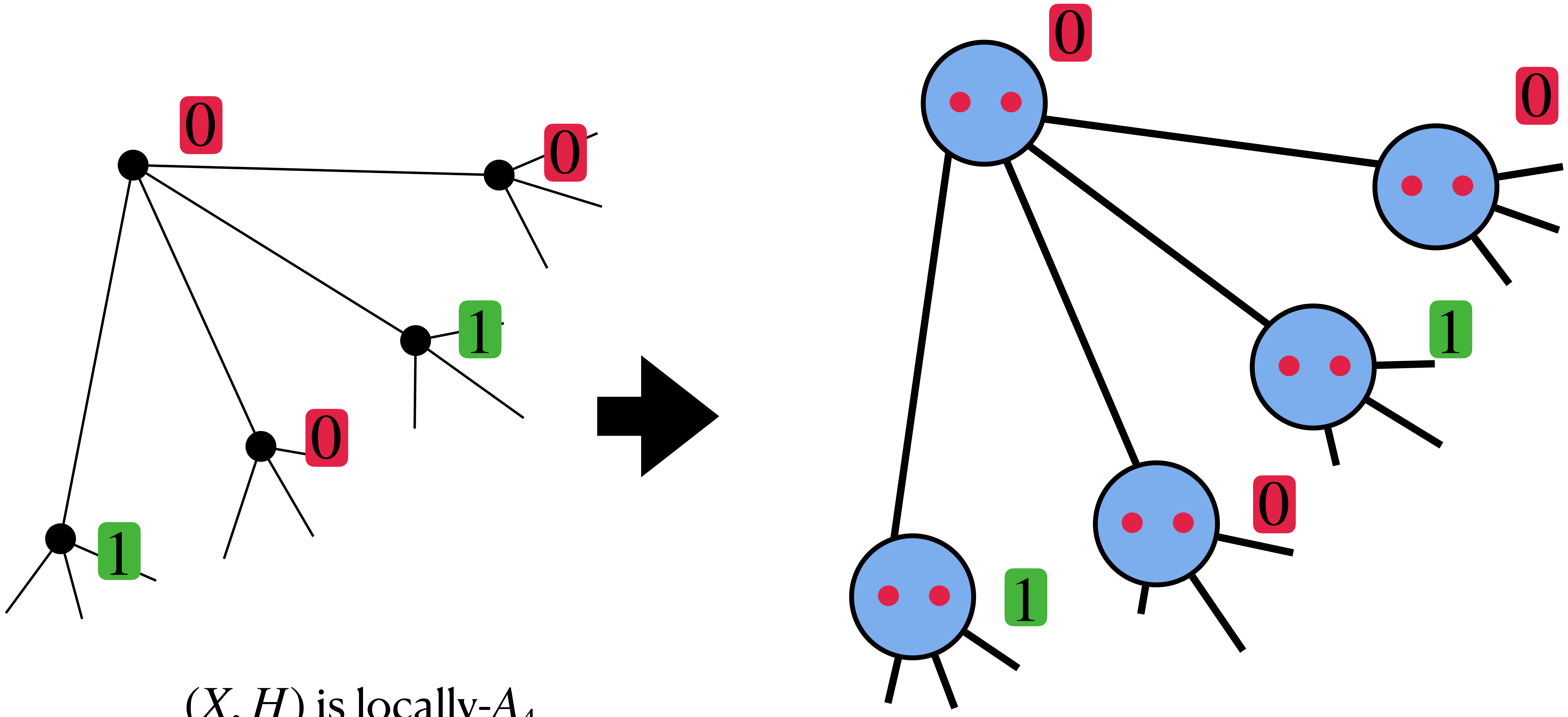

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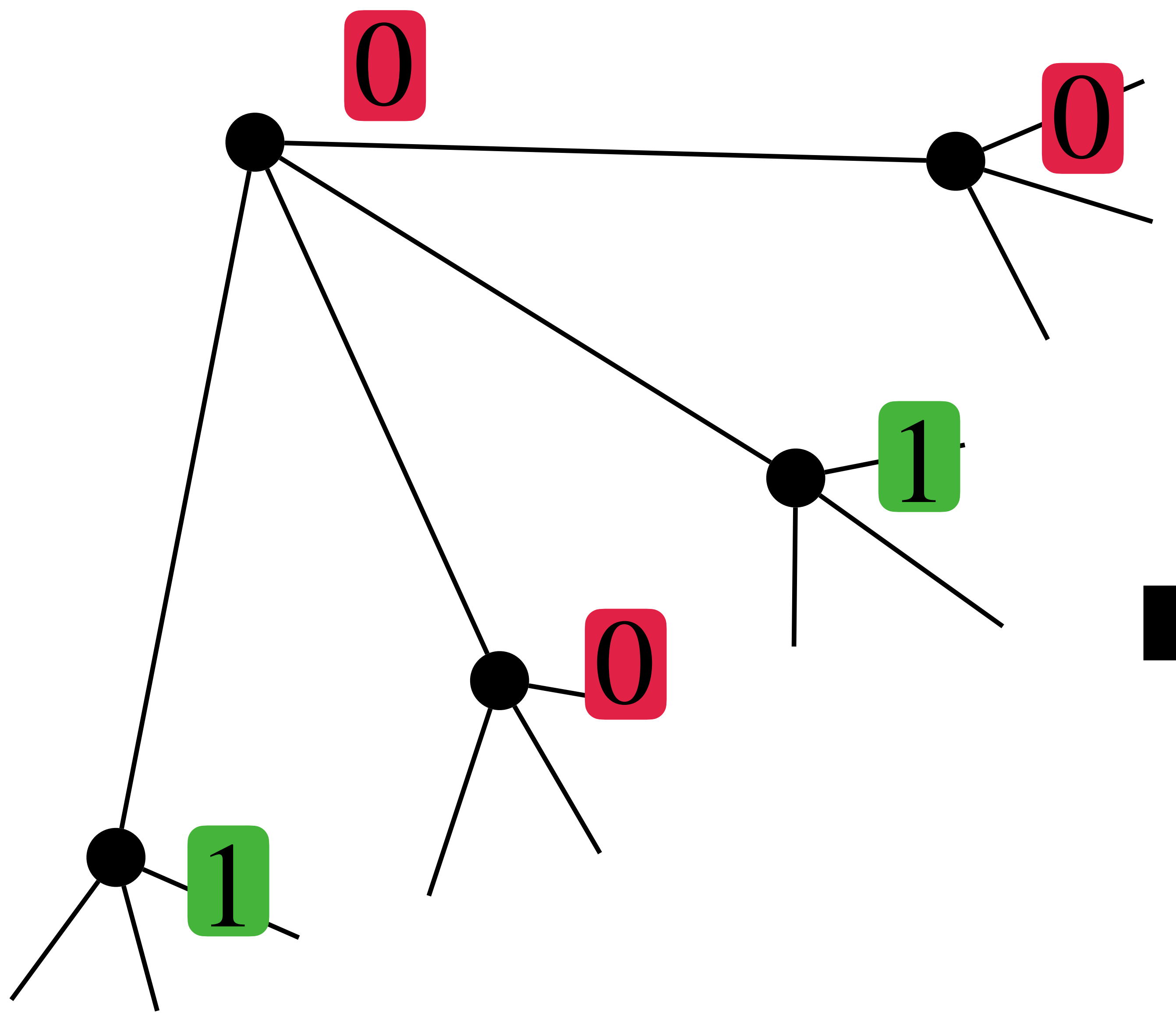

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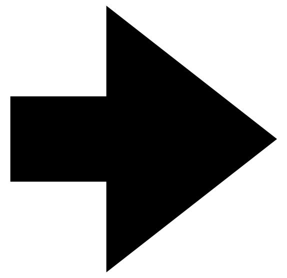

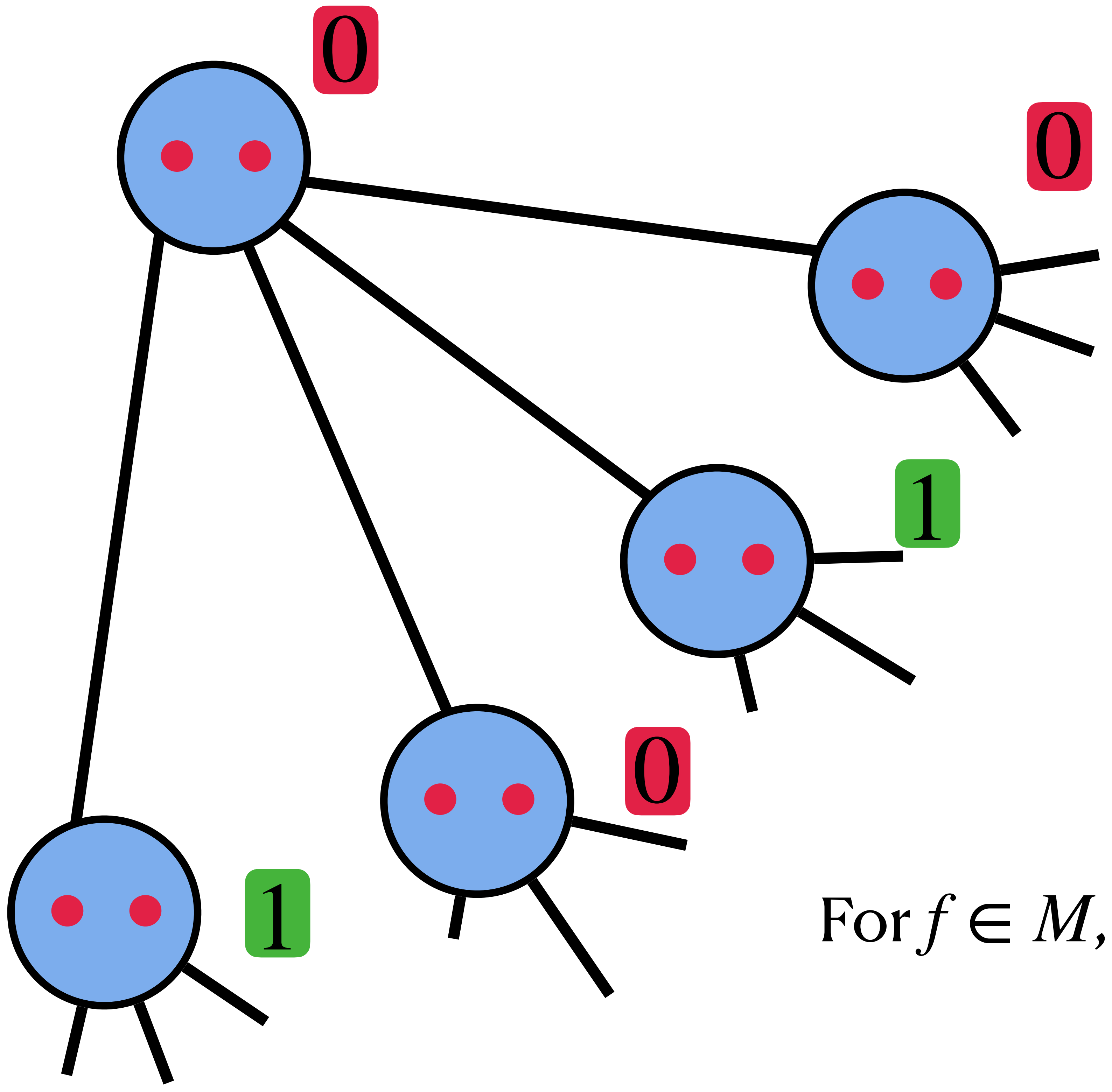

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2. $M \leq \mathbb{F}_2^{|V(X)|}$ an H-module locally inducing $V_0(\mathbb{F}_2^4)$

0-around-0 property

For $f \in M$, if $f(v) = 0$, then $\sum_{u \sim_X v} f(u) = 0$ (over $\mathbb{F}_2$).

0-around-0 modules

$X =$ vertex-transitive graph, $A =$ adjacency matrix of X over a field $\mathbb{F}$

Aut(X)-modules $M \leq \mathbb{F}^{V(X)}$ satisfying the 0-around-0 property

$$f(v) = 0 \Rightarrow (Af)(v) = \sum_{u \sim v} f(u) = 0$$

λ-eigenspaces of X

E_λ is the space of all functions $f \in \mathbb{F}^{V(X)}$ such that

$$(Af)(v) = \lambda f(v), \forall v \in V(X)$$

E_λ is a 0-around-0 module, $\forall \lambda \in \mathbb{F}$

Theorem

The non-trivial eigenspaces E_λ of X are precisely the maximal 0-around-0 modules

The same holds for general linear operators $A : \mathbb{F}^V \to \mathbb{F}^V$, where V is a transitive G-set.

Locally-$(V_0(\mathbb{F}^n) \rtimes L_1)$ pairs

via lexicographic products

$\mathscr{C} = V_0(\mathbb{F}^n) \rtimes L_1$ and (Y_m, C_m) is locally-L for all $m \in \mathbb{N}$

Construction

$$Y_m = X_m[\overline{K_q}]$$

$$C_m = \langle E_m, H_m \rangle = E_m \rtimes H_m$$

$$H_m \leq \mathrm{Aut}(X_m), E_m = E_\lambda(X_m) \text{ over } \mathbb{F}$$

Conditions

1. (X_m, H_m) is locally-L_1

4. $V_0(\mathbb{F}^n)$ is an irreducible L_1-module

ABC lemma

X = graph, $v \in V(X)$ with $C \leq A \leq \mathrm{Aut}(X)$ vertex-transitive groups of automorphisms

(X, C) is locally-$\mathscr{C}$ and (X, A) is locally-$\mathscr{A}$

Potočnik, Spiga, Verret (2014)

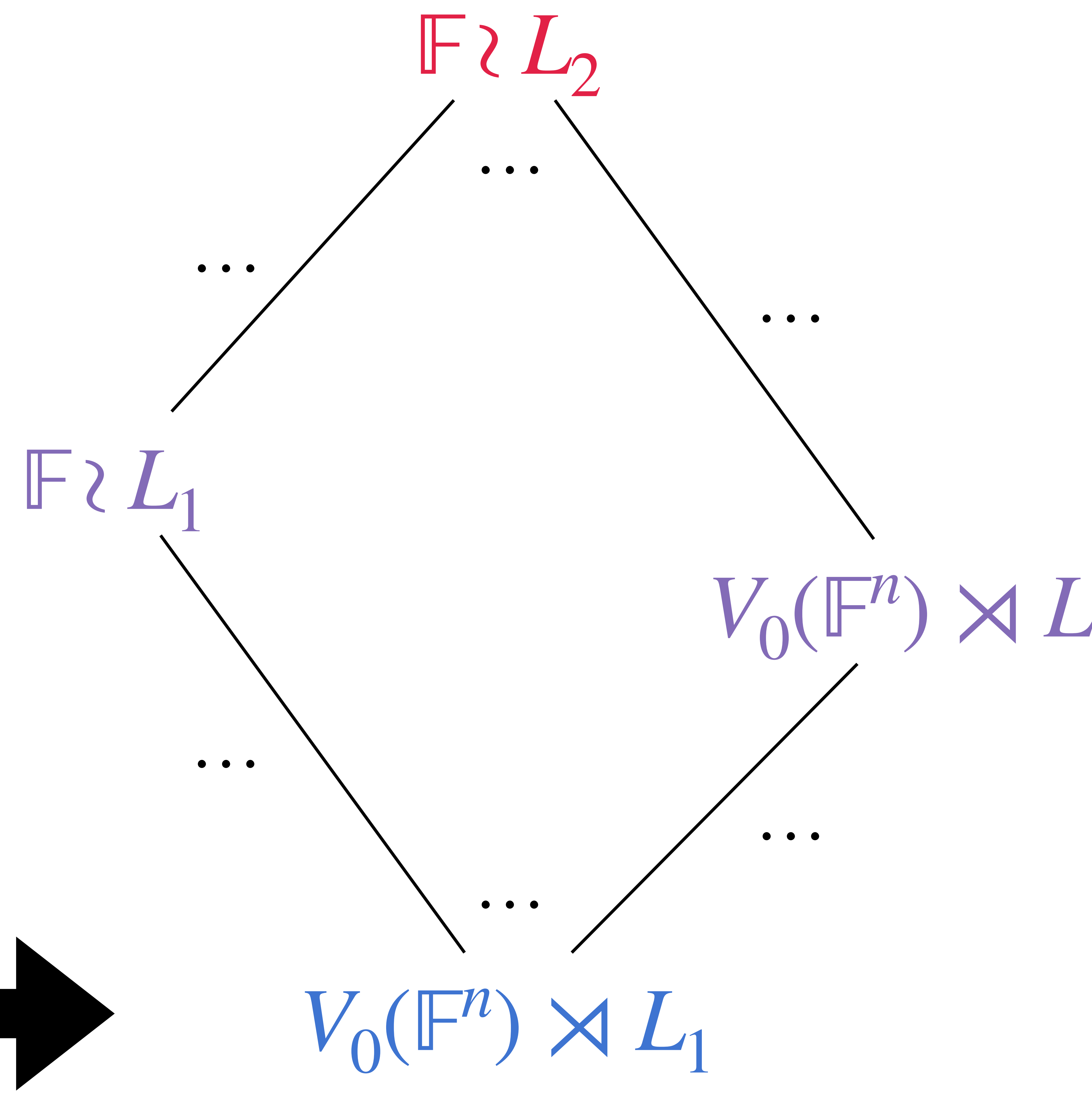
If C is **normal** in A, then for every $\mathscr{B}$ with $\mathscr{A} \geq \mathscr{B} \geq \mathscr{C}$ we can find $B \leq \mathrm{Aut}(X)$ such that:

1. $A \geq B \geq C$
2. (X, B) is locally-$\mathscr{B}$,
3. $|B_v| = t|C_v|$ with t depending only on $\mathscr{C}$.

You can get **everything** 'inbetween' A and C, and all stabilisers will have the **same growth**!!!

Applying the ABC lemma

$Y_m = X_m[\overline{K_q}]$



$C_m = \langle E_m, H_m \rangle = E_m \rtimes H_m$ with (Y_m, C_m) locally-$V_0(\mathbb{F}^n) \rtimes L_1$

Locally-$\mathbb{F} \wr L_2$ pairs

via lexicographic products

$\mathscr{A} = \mathbb{F} \wr L_2$ and (Y_m, A_m) is locally-L for all $m \in \mathbb{N}$

Construction

$$Y_m = X_m[\overline{K_q}]$$

$$A_m = \langle E_m, G_m, \lambda f_m : \lambda \in \mathbb{F} \rangle$$

$$G_m \leq \mathrm{Aut}(X_m), E_m = E_\lambda(X_m) \text{ over } \mathbb{F}$$

Conditions

2. (X_m, G_m) is locally-L_2

6. $\exists f_m \in \mathbb{F}^{V(X_m)}$ with $f_m^x - f_m \in E_\lambda(X_m), \forall x \in H_m$

Applying the ABC lemma

$Y_m = X_m[\overline{K_q}]$

$A_m = \langle E_m, G_m, \lambda f_m : \lambda \in \mathbb{F} \rangle$ with (Y_m, A_m) locally-$\mathbb{F} \wr L_2$

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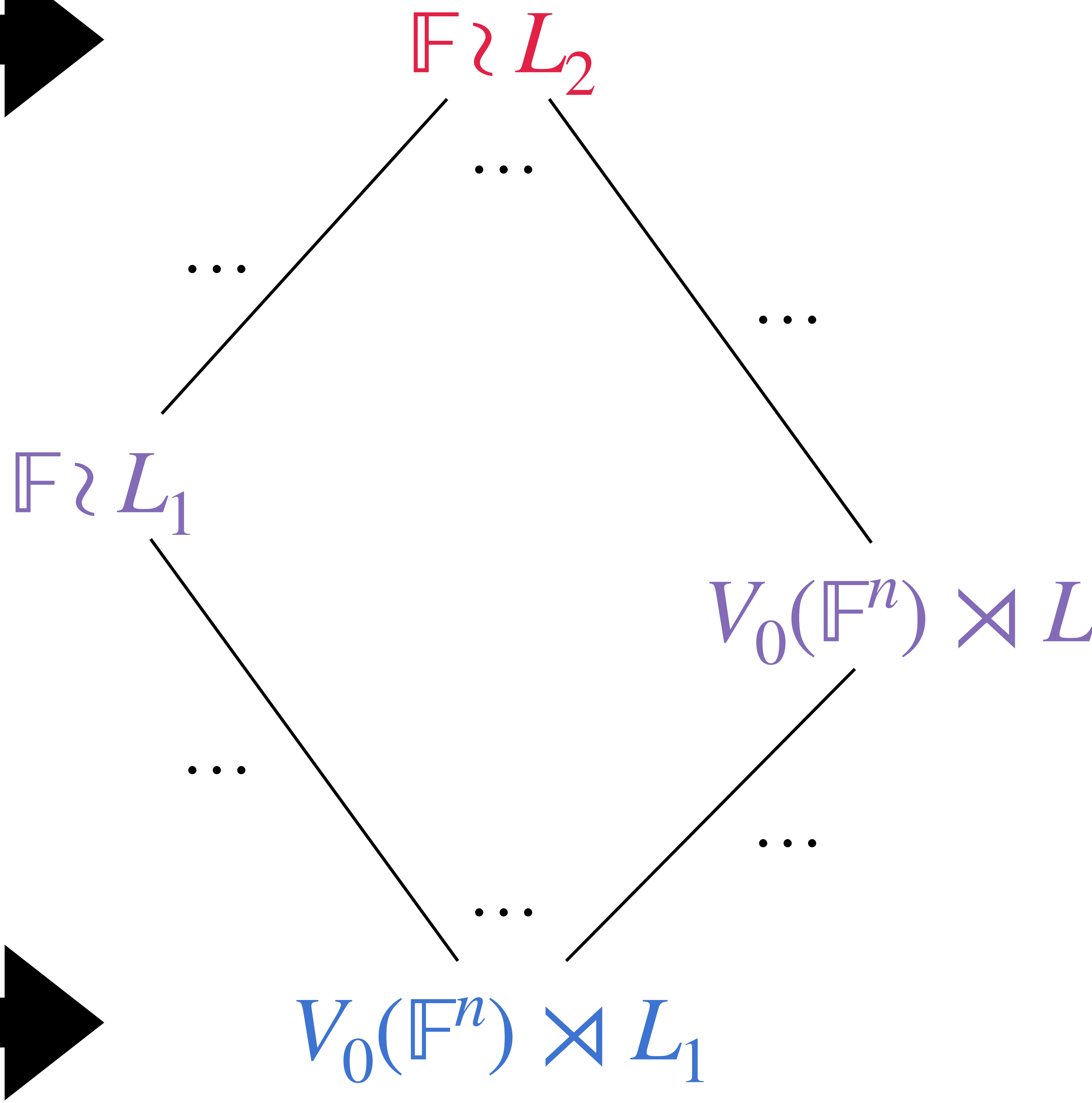

Applying the ABC lemma

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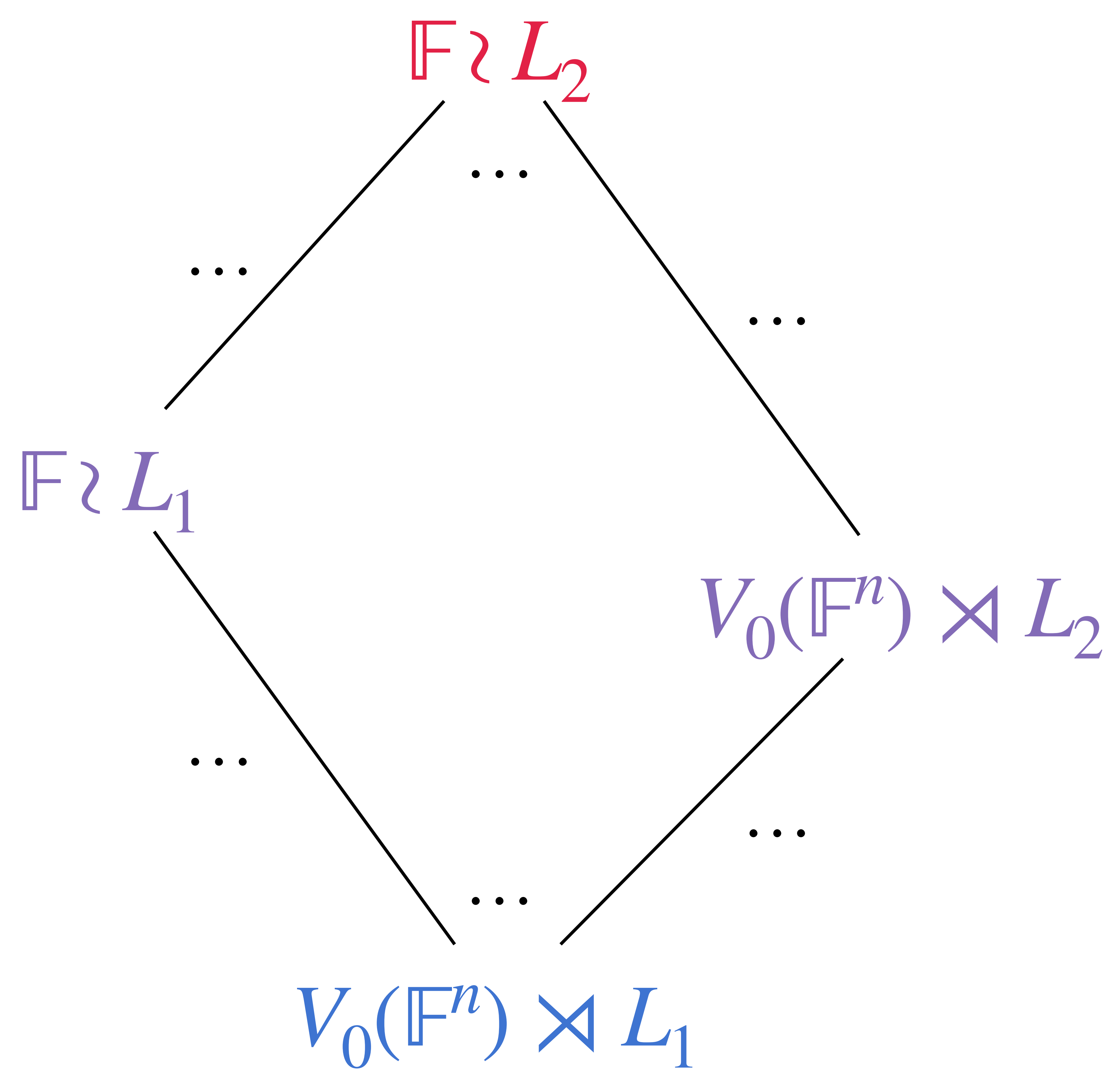
$C_m = \left\langle E_m, H_m \right\rangle = E_m \rtimes H_m$ with (Y_m, B_m) locally-$V_0(\mathbb{F}^n) \rtimes L_1$

C_m is normal in A_m

Conditions

3. H_m is normal in G_m

6. $\exists f_m \in \mathbb{F}^{V(X_m)}$ with $f_m^x - f_m \in E_m, \forall x \in H_m$



ABC Lemma:

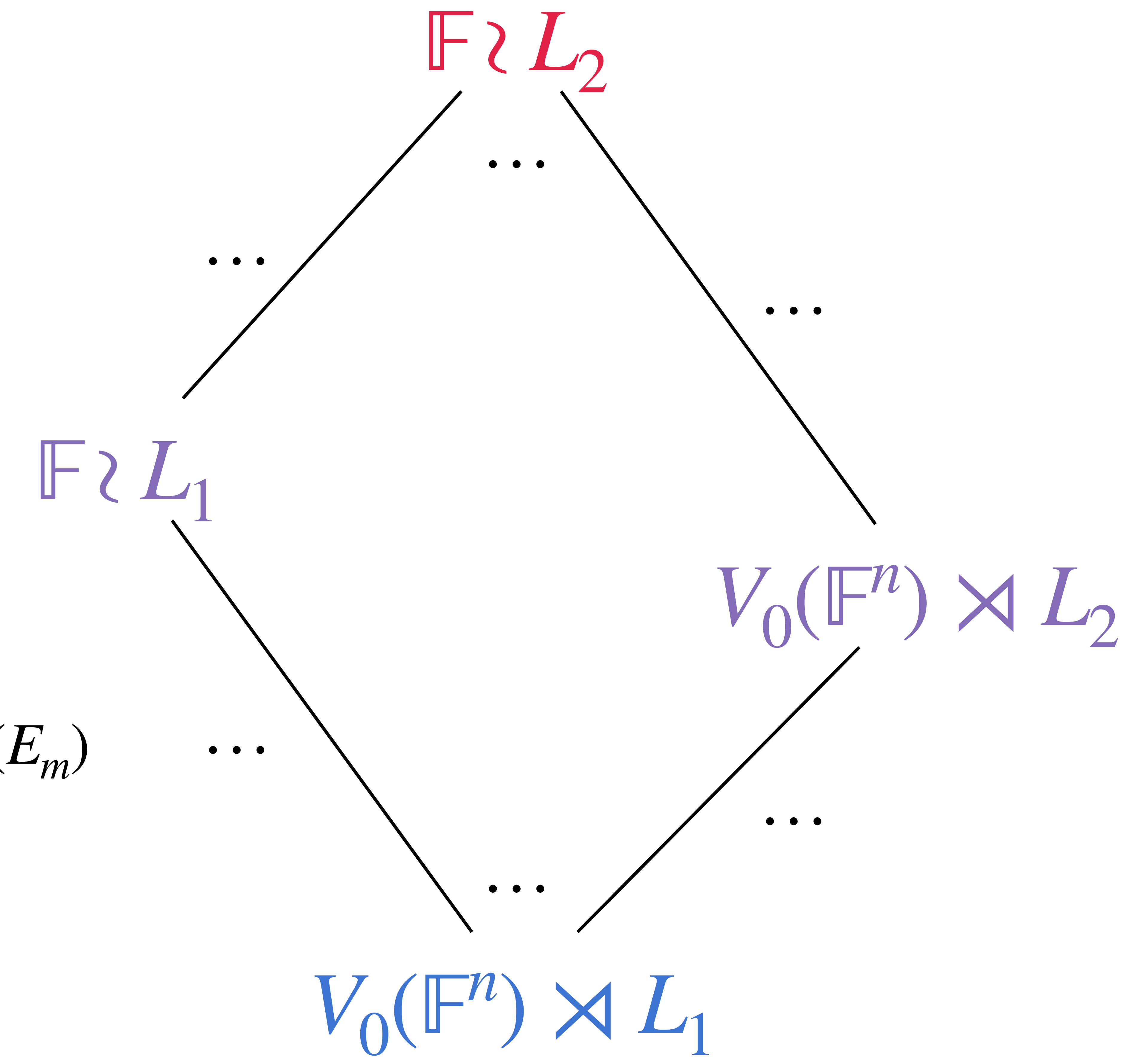
For all L with $V_0(\mathbb{F}^n) \rtimes L_1 \leq L \leq \mathbb{F} \wr L_2$

we obtain locally-L pairs $\{(Y_m, B_m)\}_{m\in\mathbb{N}}$ with

$$|(B_m)_v| = t|(C_m)_v|$$

$$|(B_m)_v| = t|(C_m)_v| = t|(E_m \rtimes H_m)_v| \geq |E_m| = q^{\dim(E_m)}$$

$$\dim_{\mathbb{F}}(E_\lambda(X_m)) \geq c|V(X_m)|$$



$$|(B_m)_v| = t|(C_m)_v| = t|(E_m \rtimes H_m)_v| \geq |E_m| = q^{\dim(E_m)}$$

$$\dim_{\mathbb{F}}(E_\lambda(X_m)) \geq c\,|V(X_m)|$$

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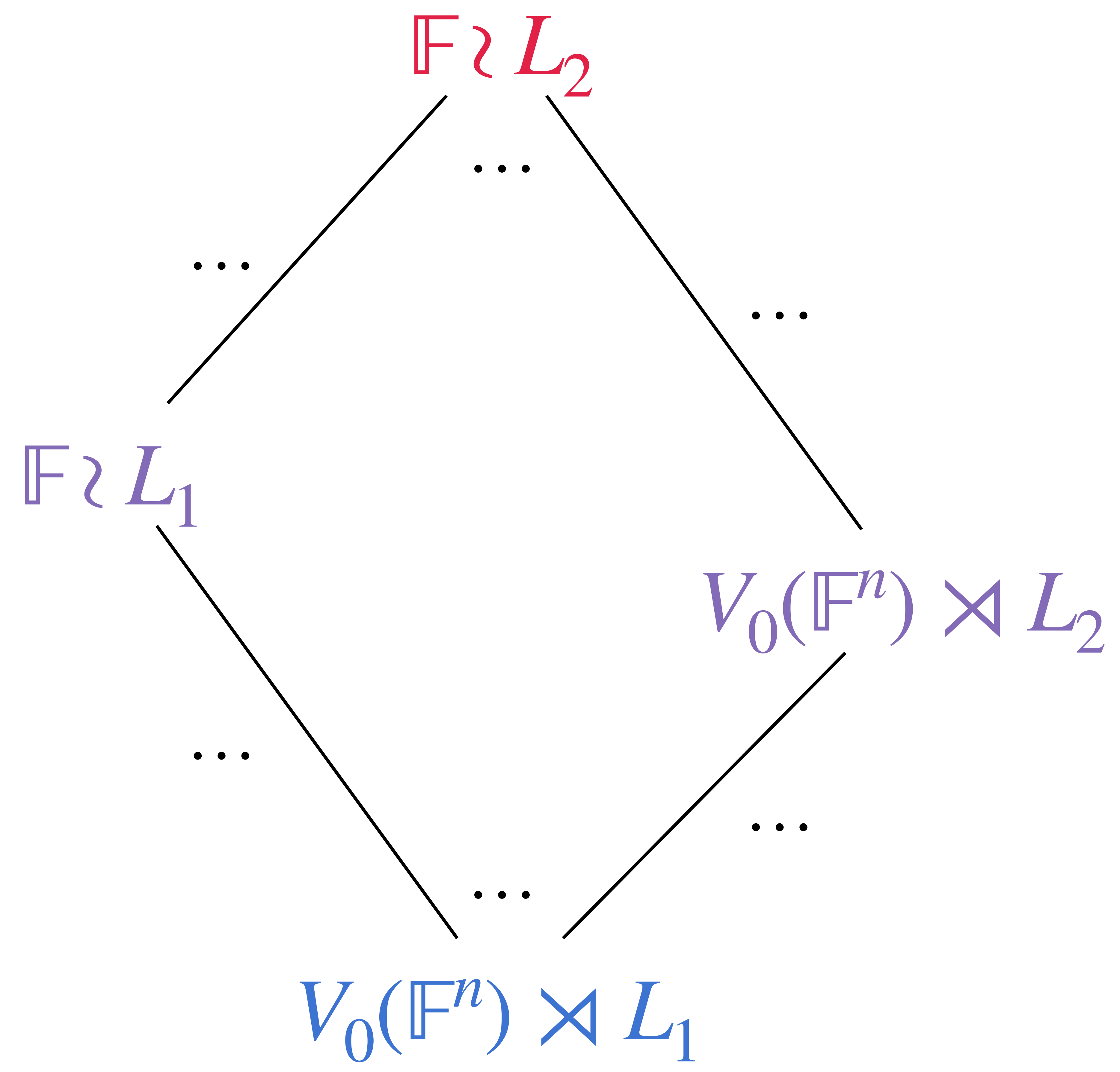
we obtain locally-L pairs $\{(Y_m, B_m)\}_{m \in \mathbb{N}}$ with

$$|(B_m)_v| \geq q^{\dim(E_m)} \geq (q^{c/q})^{|V(Y_m)|}$$



EXPONENTIAL GROWTH!

$L_1 \leq L_2$ = transitive permutation groups of degree n

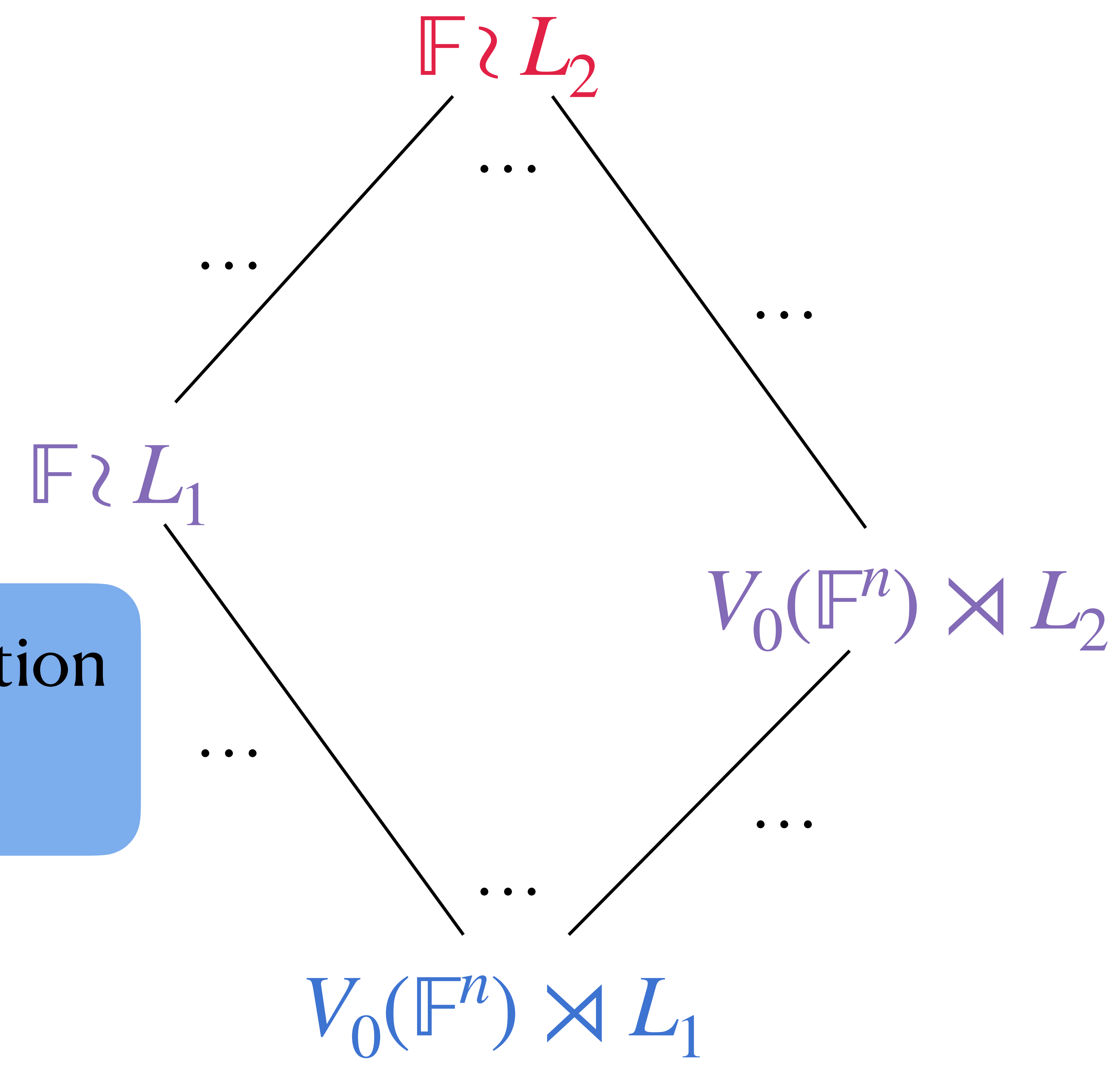
$\mathbb{F}$ = finite field of order q

$\{X_m\}_{m\in\mathbb{N}}$ = infinite family of n-regular graphs

$H_m, G_m \leq \mathrm{Aut}(X_m)$ = vertex-transitive groups of symmetries

1. (X_m, H_m) is locally-L_1,
2. (X_m, G_m) is locally-L_2,
3. H_m is normal in G_m,
4. $V_0(\mathbb{F}^n)$ is an irreducible L_1-module,
5. $\exists \lambda \in \mathbb{F}$ and a constant $c > 0$ such that $\dim_{\mathbb{F}}(E_\lambda(X_m)) \geq c\,|V(X_m)|$,
6. $\exists f_m \in \mathbb{F}^{V(X_m)}$ such that $f_m^x - f_m \in E_\lambda(X_m), \forall x \in H_m$.

Ensuring the correct local action on the bottom



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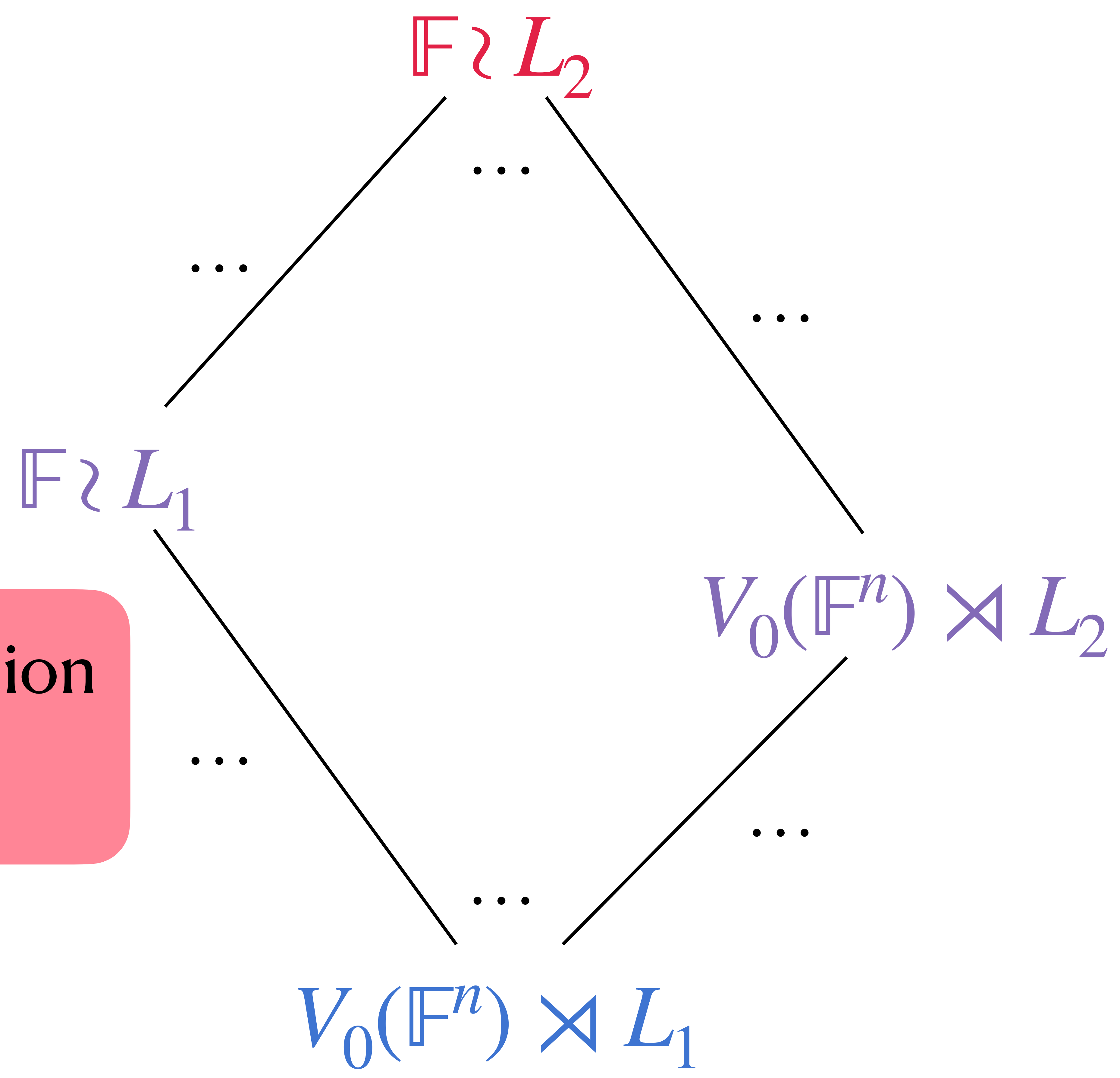
Ensuring the correct local action on the top



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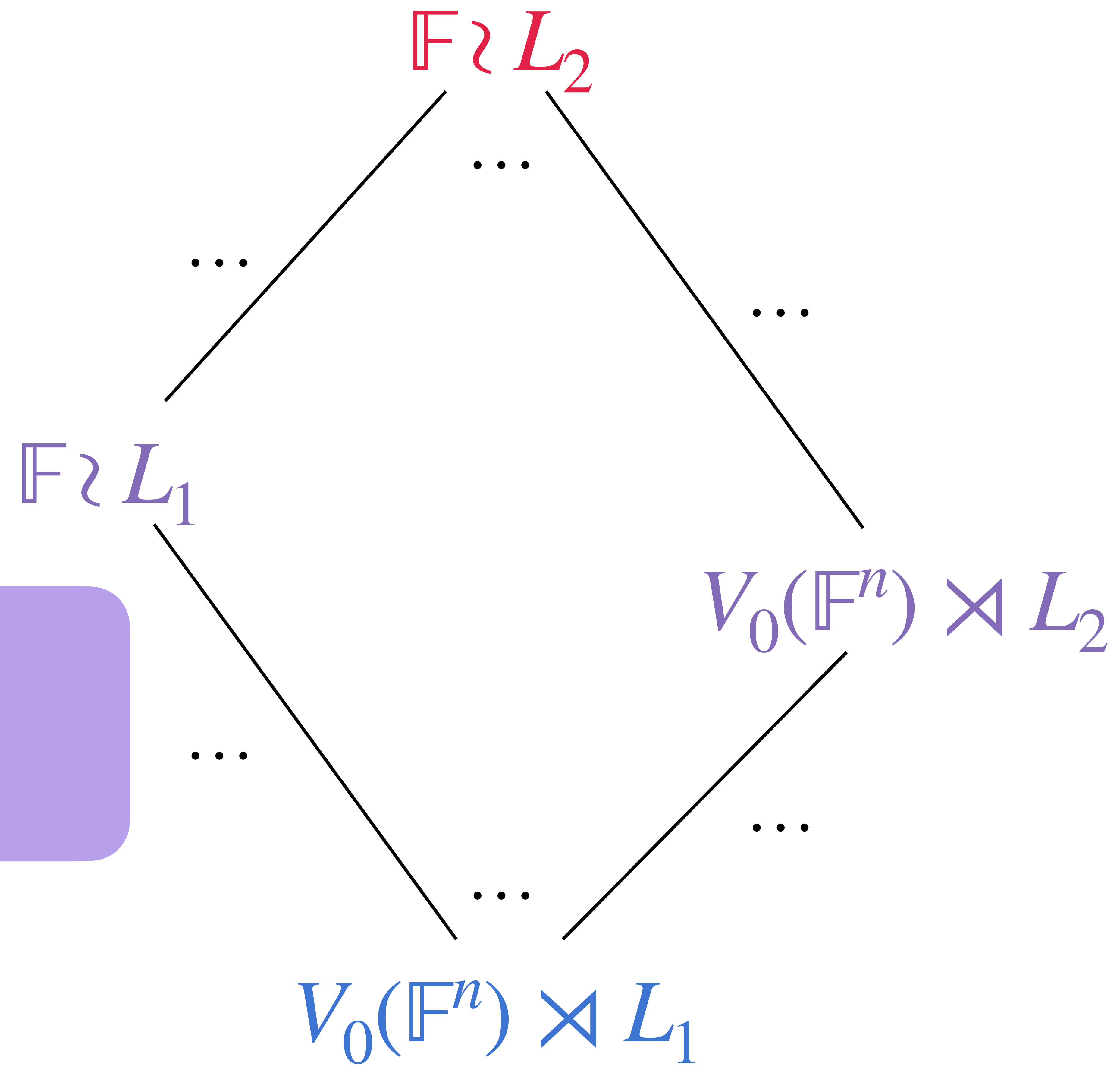
Ensuring we can apply the ABC lemma



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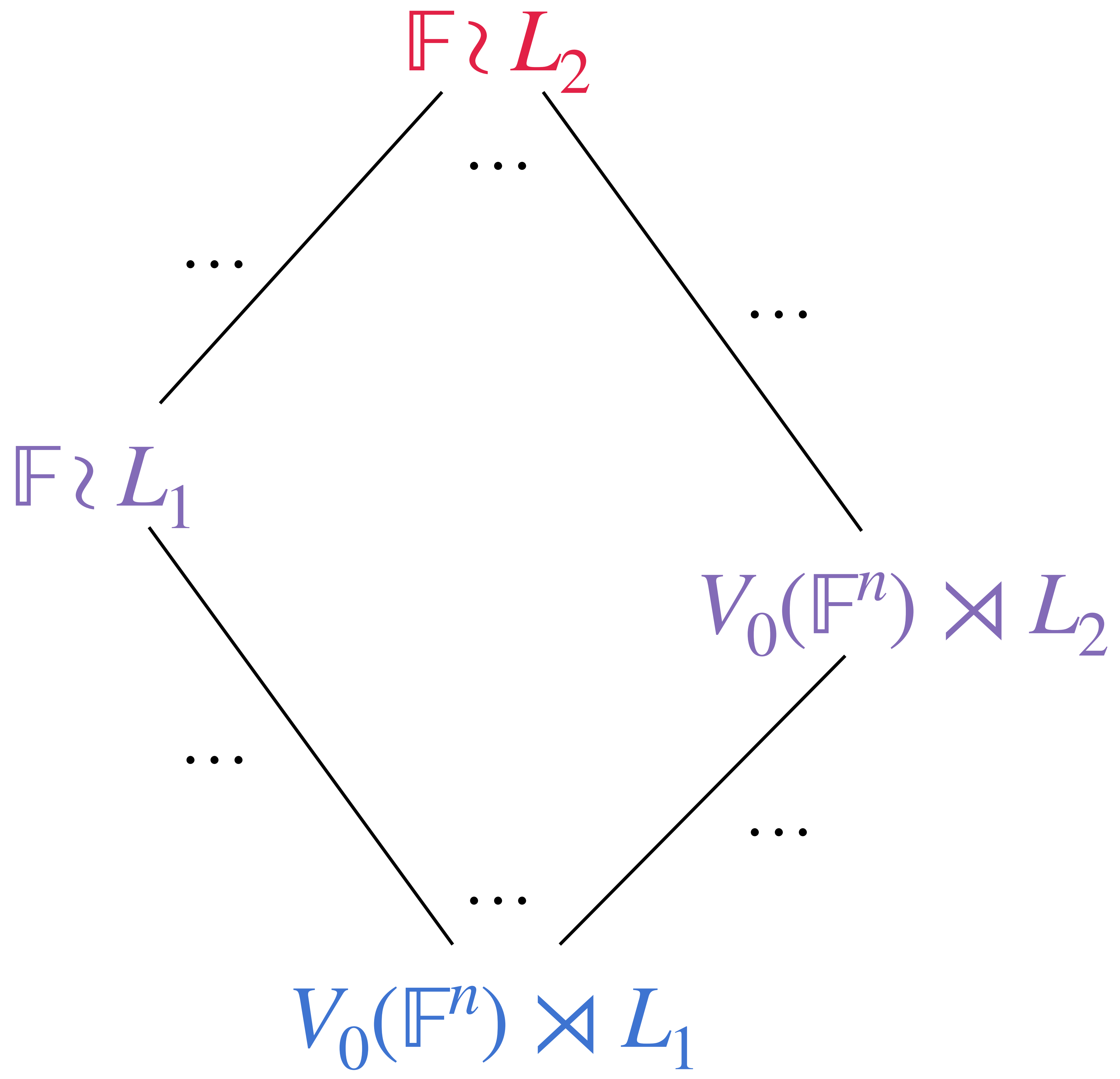

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GROWTH!

$L_1 \leq L_2$ = transitive permutation groups of degree n

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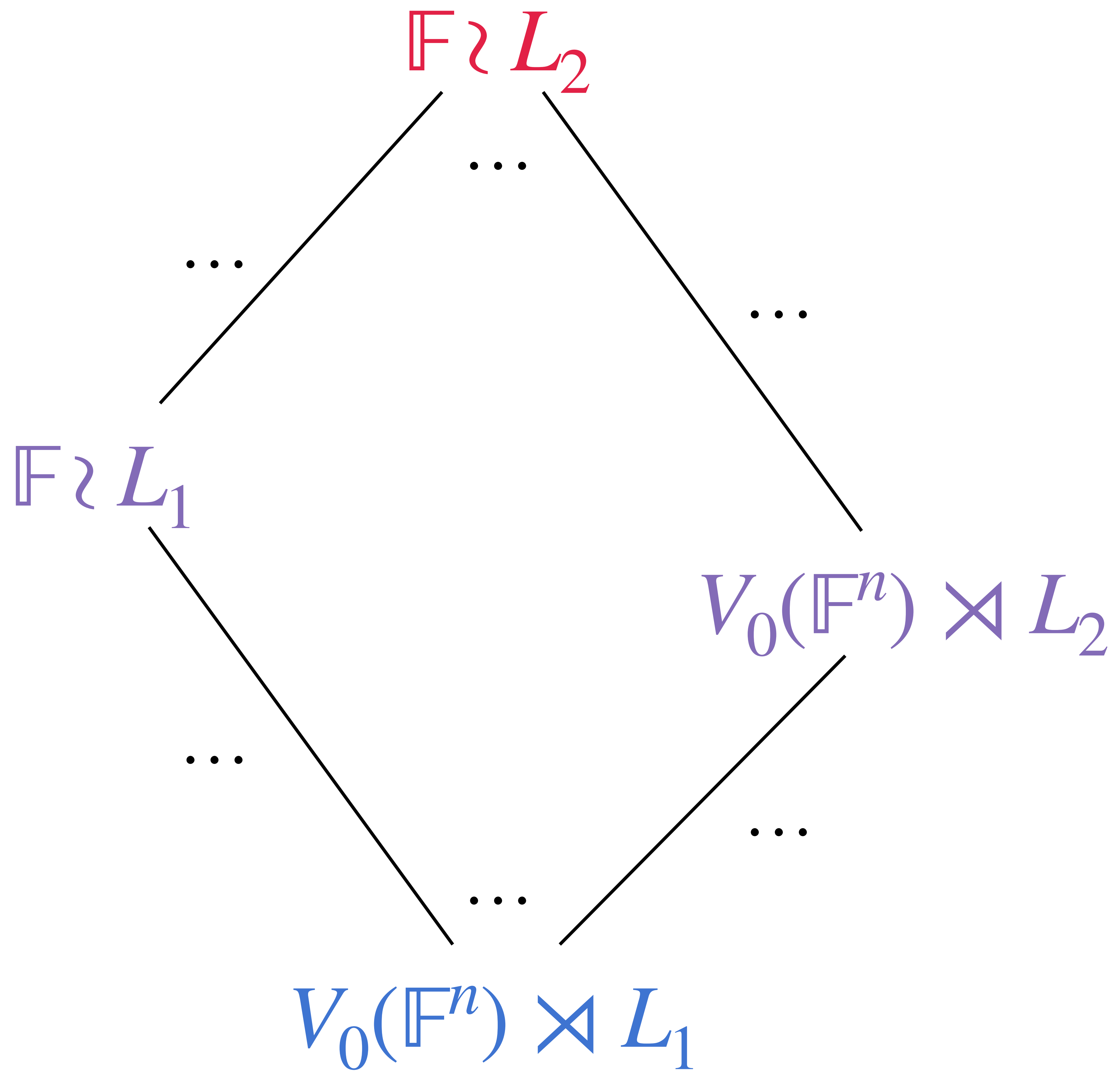

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Every group L with $V_0(\mathbb{F}^n) \rtimes L_1 \leq L \leq \mathbb{F} \wr L_2$ has exponential graph growth.

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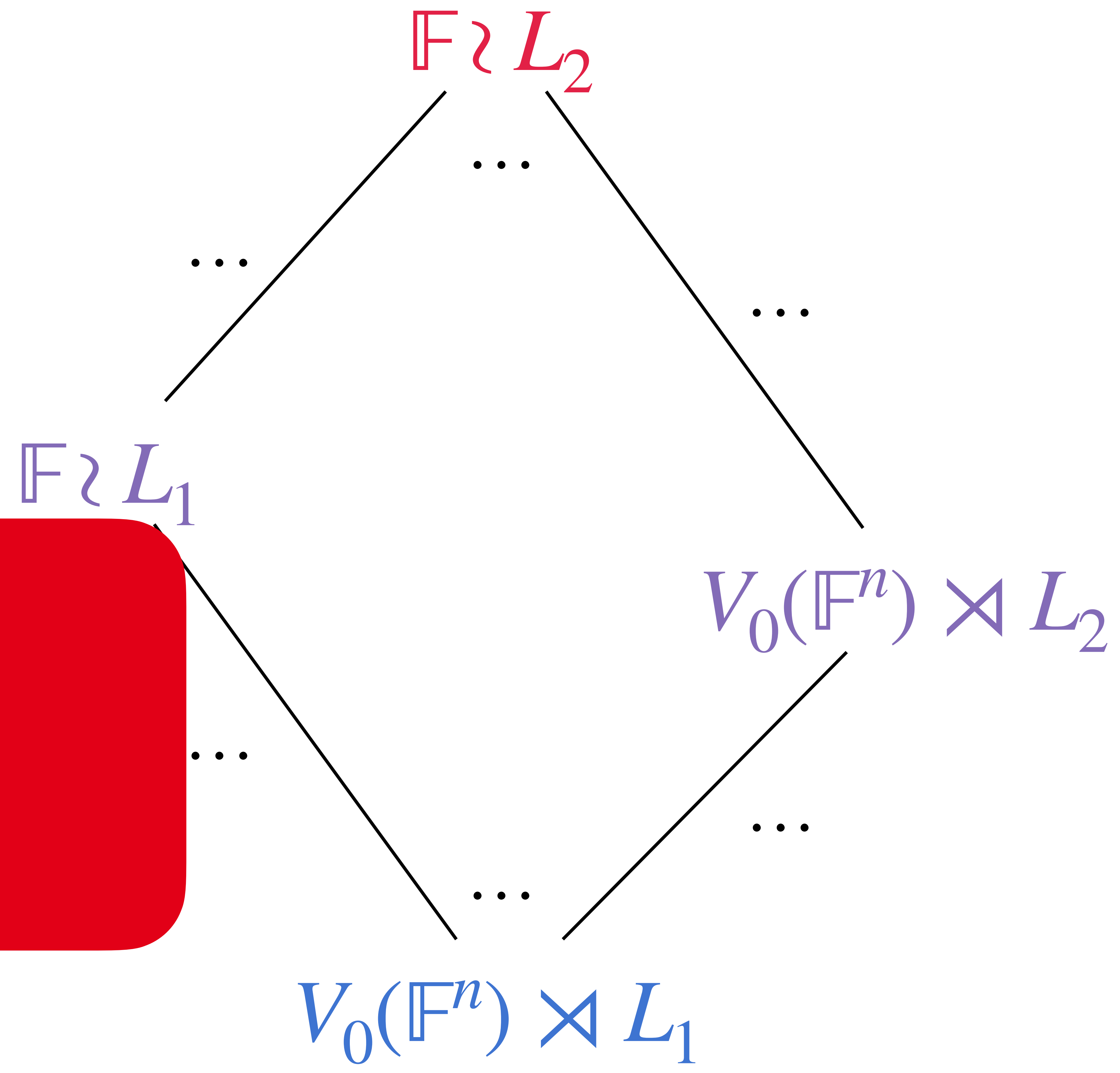

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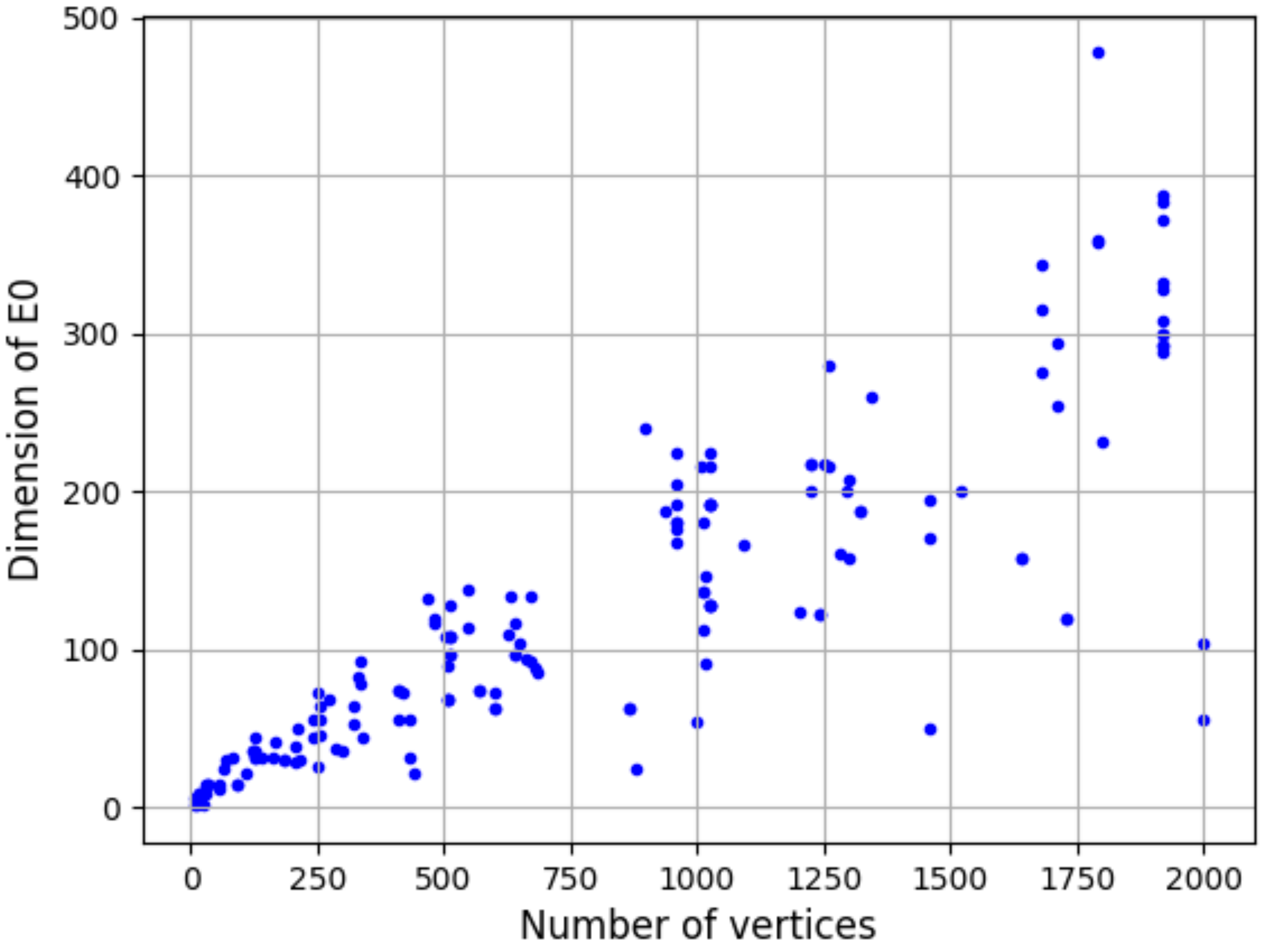
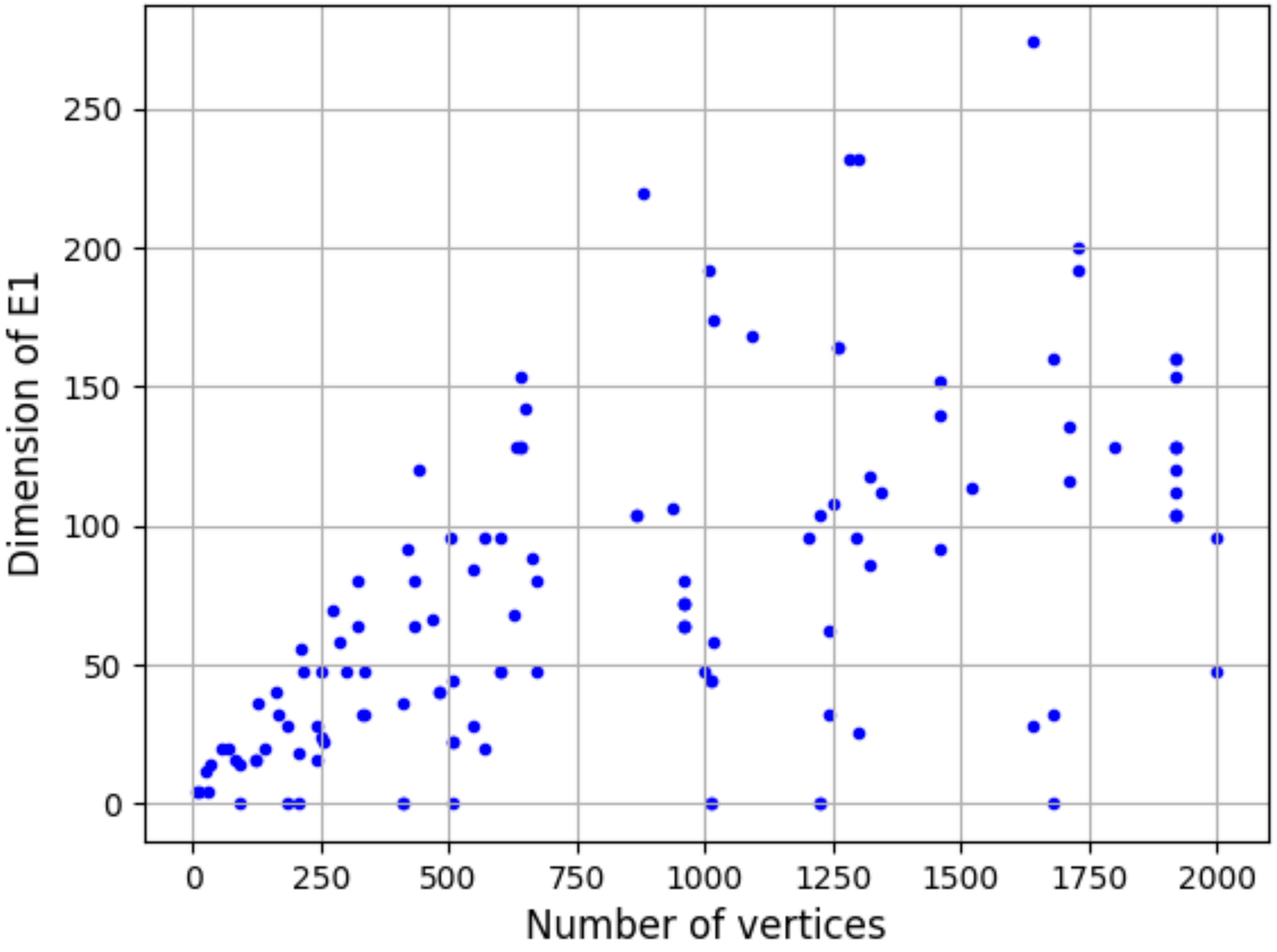
How do we find $\{X_m\}_{m \in \mathbb{N}}$?

Every group L with $V_0(\mathbb{F}^n) \rtimes L_1 \leq L \leq \mathbb{F} \wr L_2$ has exponential graph growth.

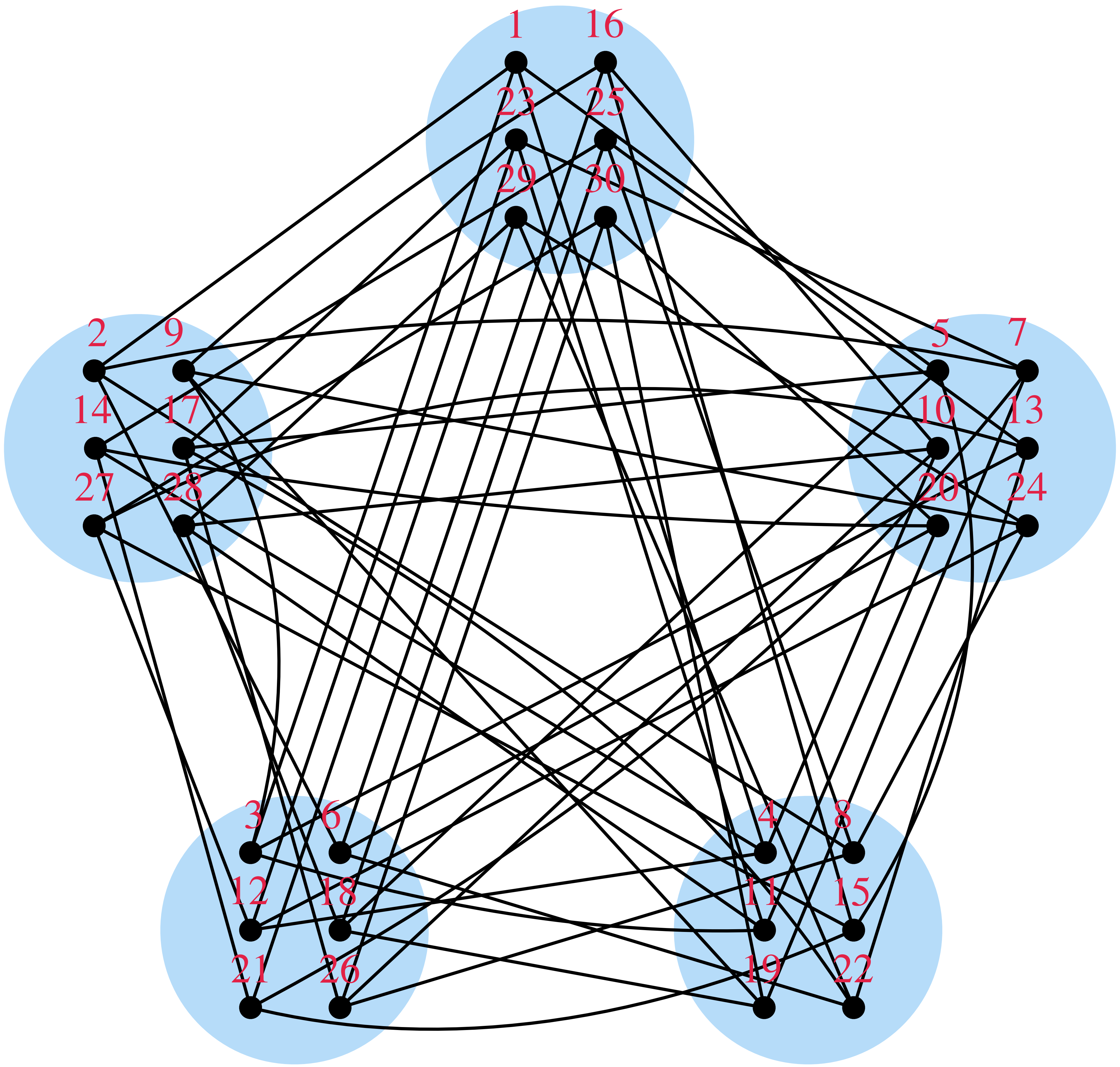
$\mathbb{F}_2$-eigenspaces of 4-valent 2-arc-transitive graphs

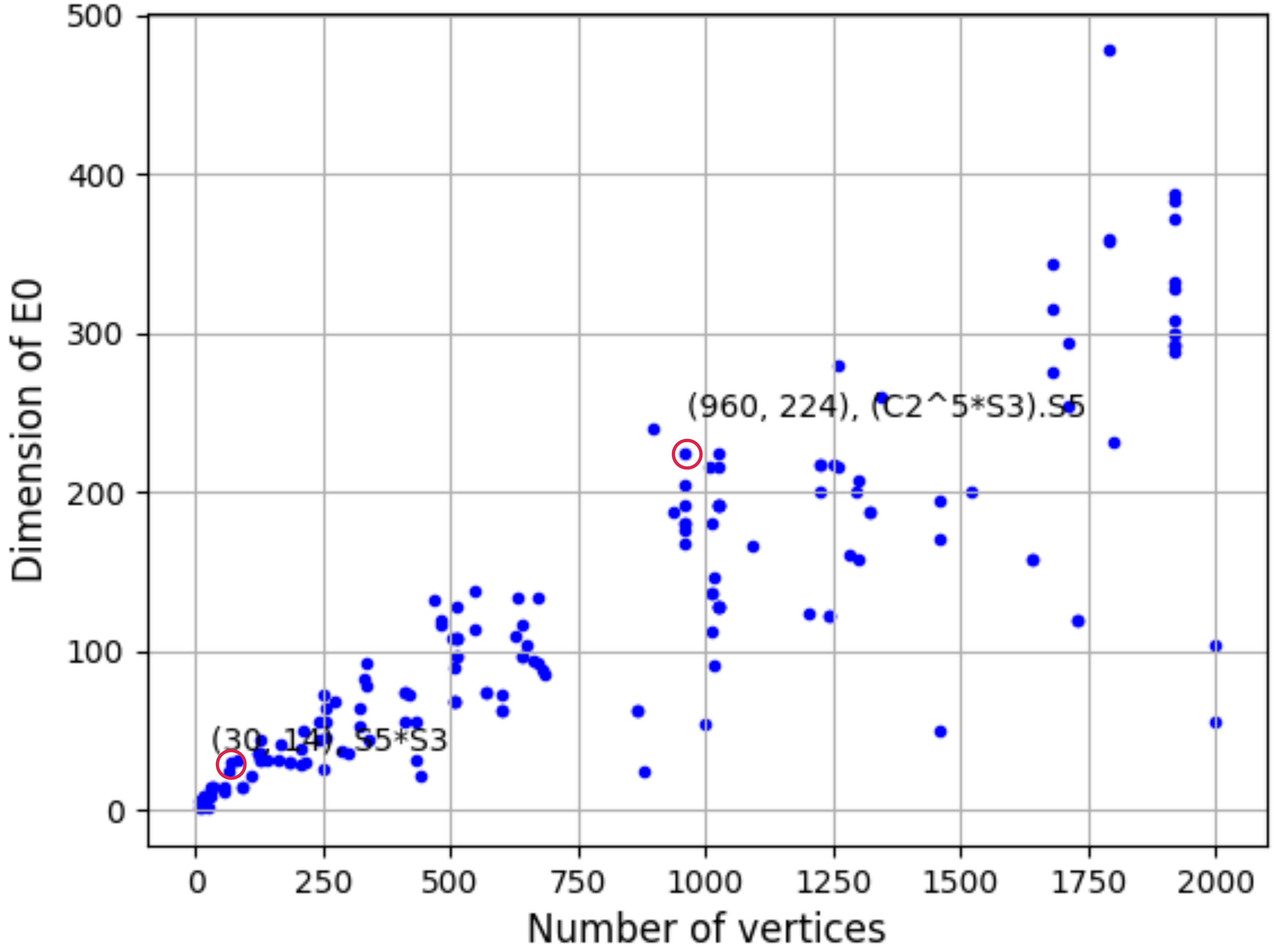
Database of all locally-A_4 and locally-S_4 graphs on at most 2000 vertices (Potočnik, 2008)





$\mathbb{Z}_m^5$– voltage covers of T_{30}





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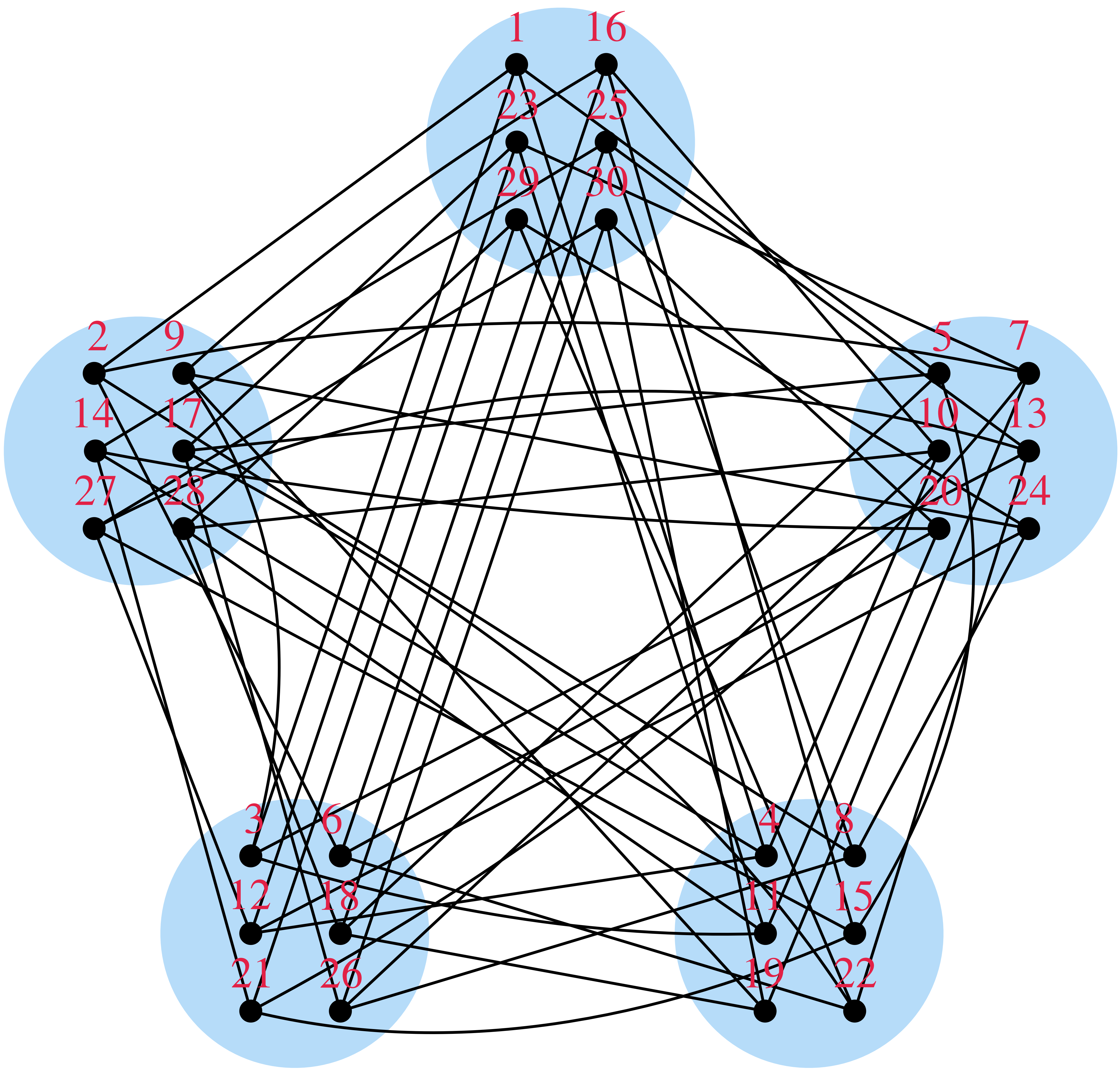


Method

Malnič, Marušič and Potočnik (2004)

- (T_{30}, H) is locally-A_4 and (T_{30}, G) is locally-S_4
- Basis of a 5-dimensional subspace $U \leq H_1(T_{30}, \mathbb{Z}_m)$ invariant under $\mathrm{Aut}(T_{30}) \curvearrowright H_1(T_{30}, \mathbb{Z}_m)$

Result

$X_m = \mathbb{Z}_m^5$-cover of T_{30}

(X_m, H_m) is locally-A_4 and (X_m, G_m) is locally-S_4

H_m is normal in G_m

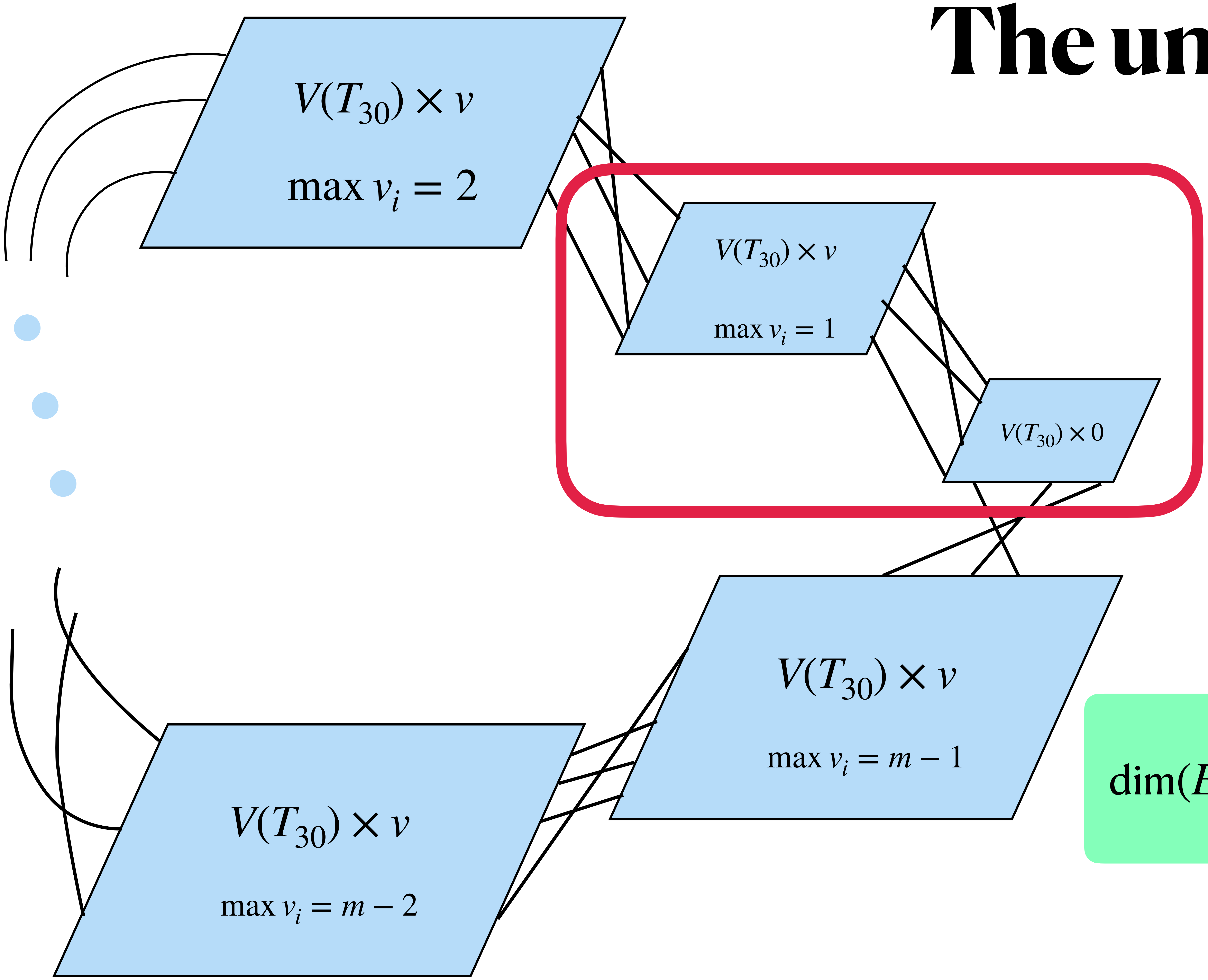
What about the $\mathbb{F}_2$-eigenspaces of X_m?

m	$\lvert V(X_m)\rvert / \dim(E_0(X_m))$	$\lvert V(X_m)\rvert / \dim(E_1(X_m))$
1	2.142	7.5
2	4.285	12
3	4.972	18.044
4	4.897	23.237
5	4.997	20.907
6	4.996	35.132

Conjecture:

$$\dim(E_0(X_m)) = \begin{cases} \frac{\lvert V(X_m)\rvert}{5} + 8, \text{ when } m \text{ is odd} \\ \frac{\lvert V(X_m)\rvert}{5} + 32, \text{ when } m \text{ is even} \end{cases}$$

The universal eigenvector

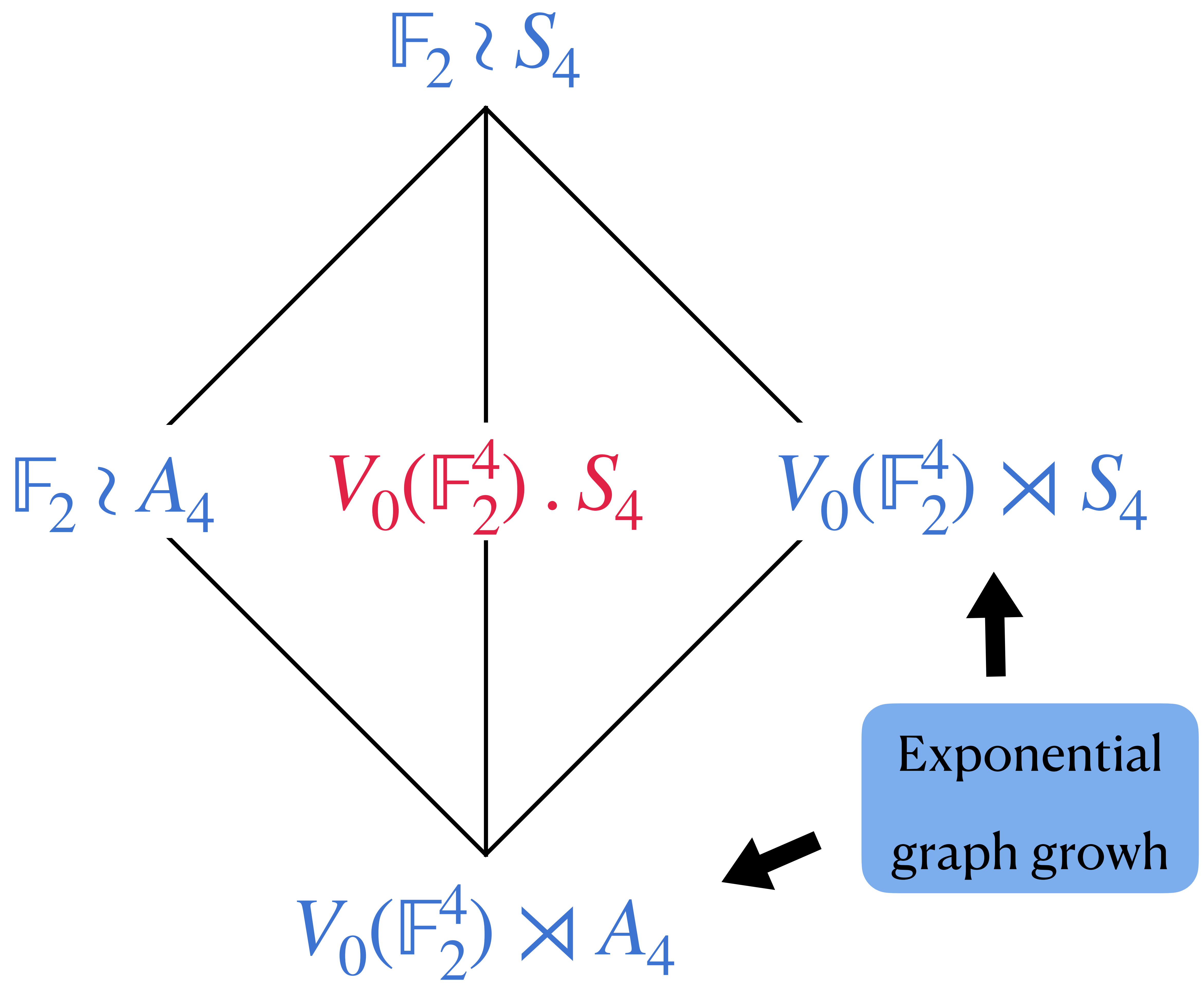


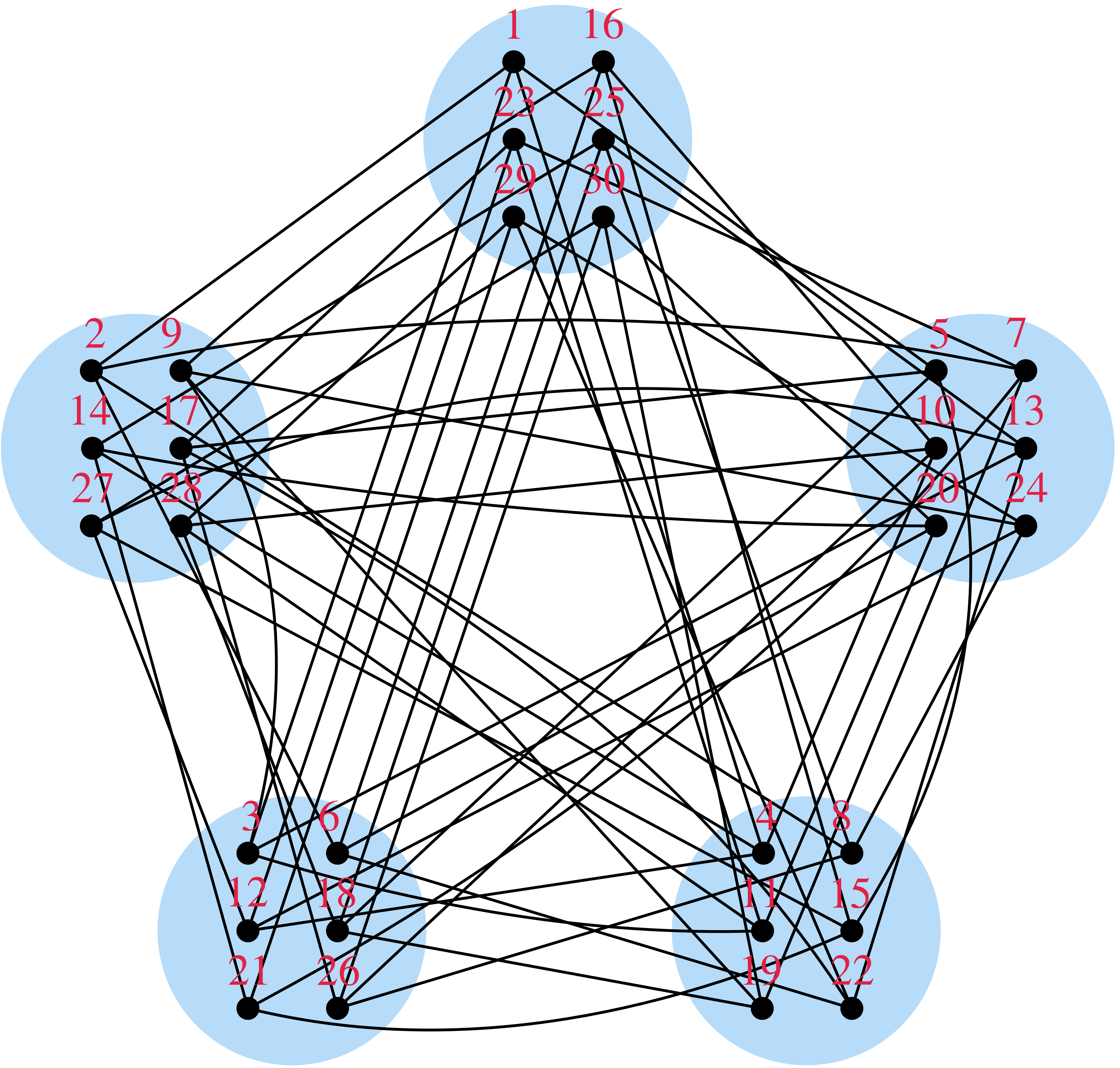
Using **MAGMA** we obtain:

$x =$ 0-eigenvector of X_m with support of size 15 located in this subset

$$\dim(E_0(X_m)) \geq \dim \left\langle x^g : g \in \mathrm{Aut}(X_m) \right\rangle \geq \frac{|V(X_m)|}{15}$$

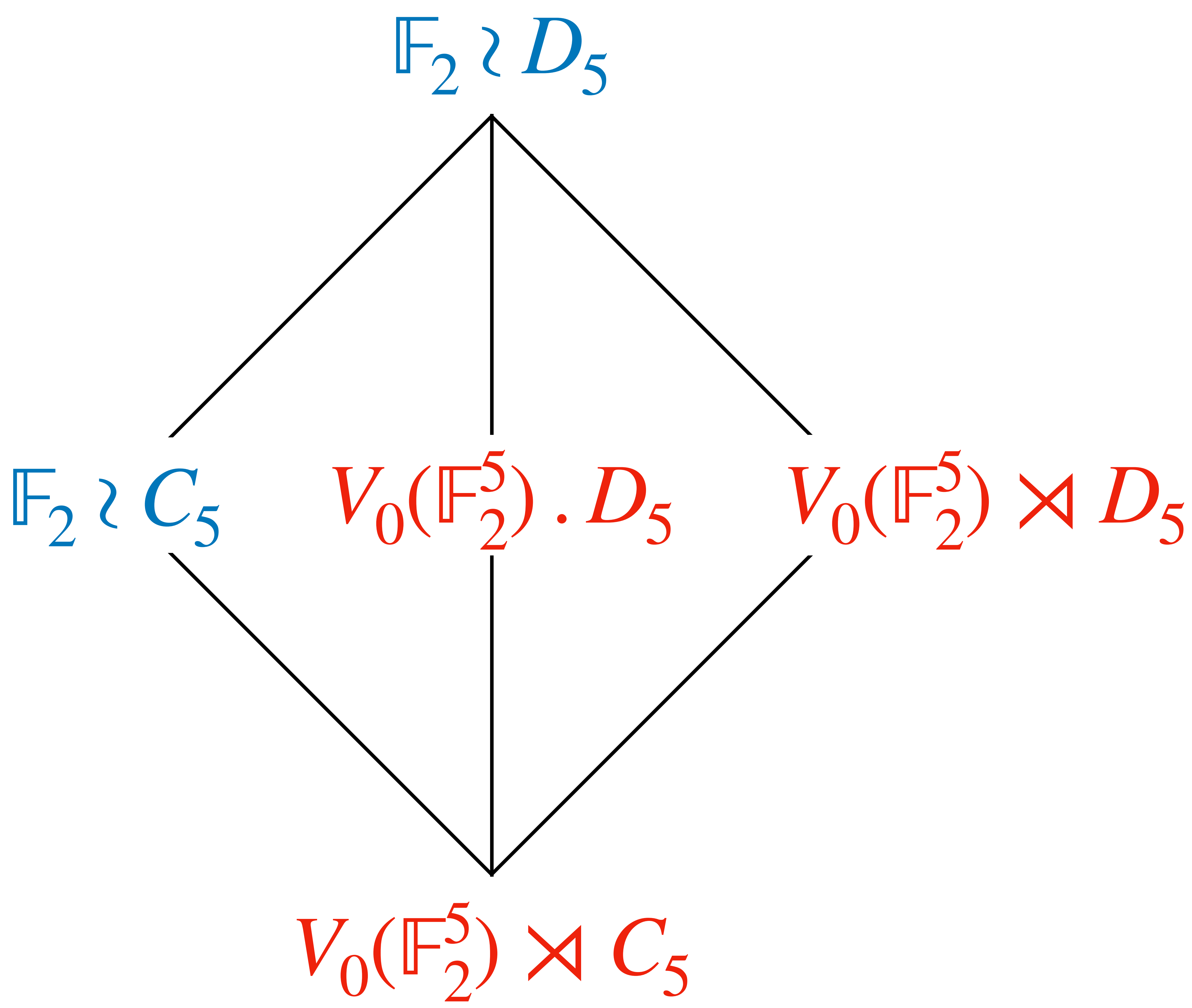
Degree 8

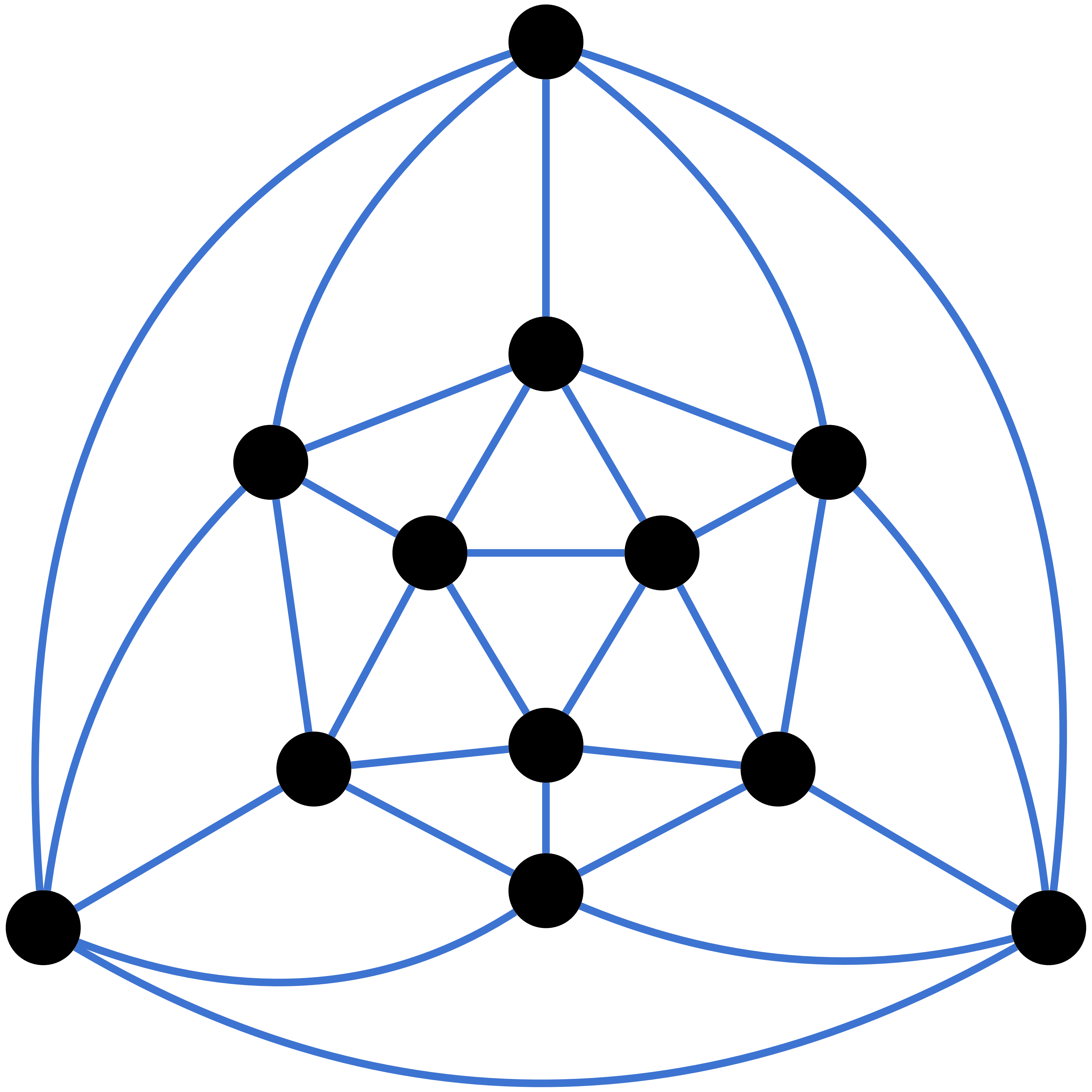


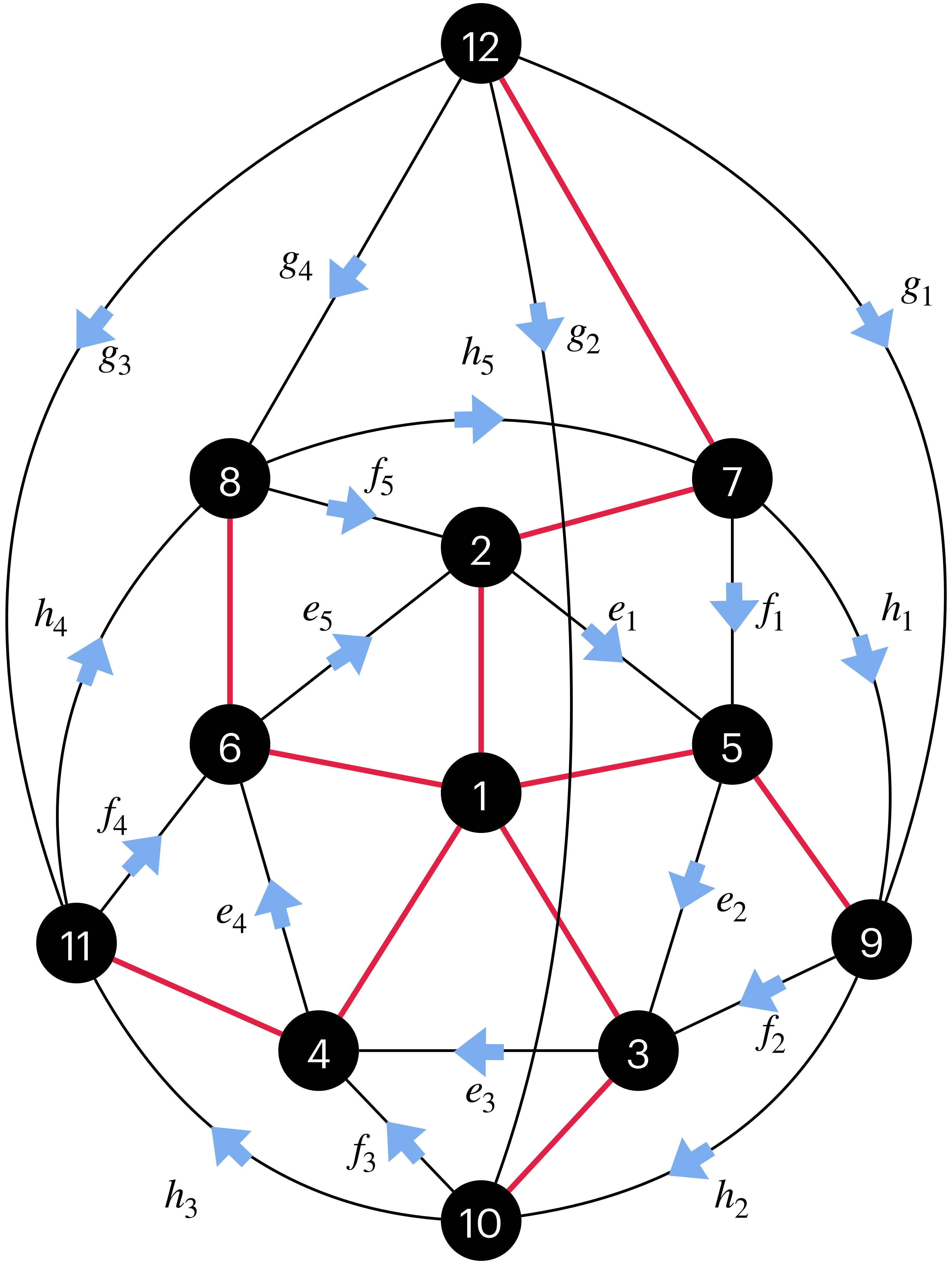


$\exists f_m \in \mathbb{F}^{V(X_m)}$ such that $f_m^x - f_m \in E_\lambda(X_m), \forall x \in H_m$

Graph growth of some groups of degree 10







Method

Malnič, Marušič and Potočnik (2004)

- (▯, H) is locally-C_5 and (▯, G) is locally-D_5 with H normal in G
- Basis of a 4-dimensional subspace $U \leq H_1($▯$, \mathbb{Z}_m)$ invariant under Aut(▯) $\curvearrowright H_1($▯$, \mathbb{Z}_m)$

Result

$I_m = \ \mathbb{Z}_m^4$-cover of ▯

(I_m, H_m) is locally-C_5 and (I_m, G_m) is locally-D_5

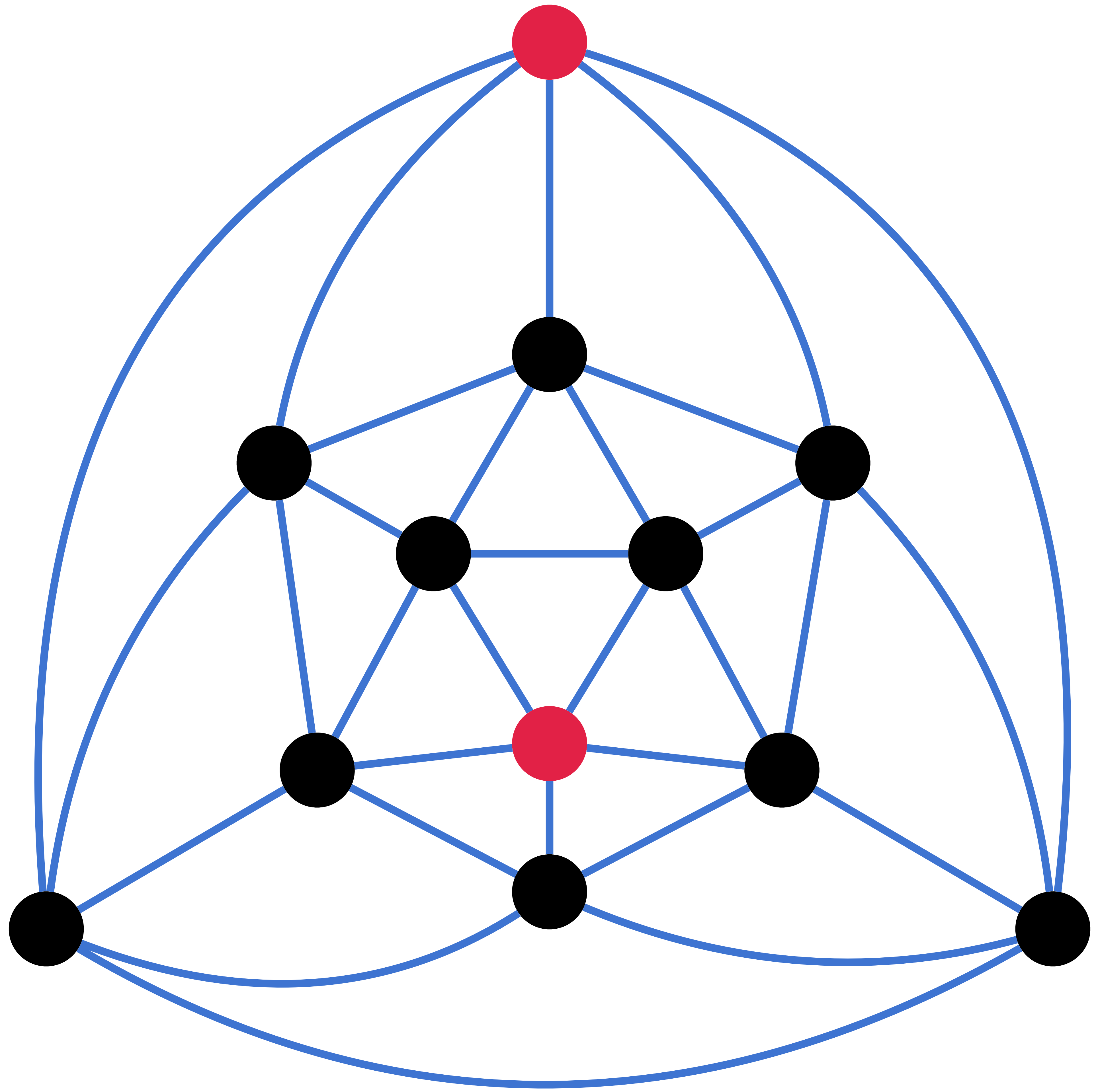
H_m is normal in G_m

$\mathbb{F}_2$-eigenspaces of I_m

m	$\lvert V(I_m)\rvert / \dim(E_0(I_m))$	$\lvert V(I_m)\rvert / \dim(E_1(I_m))$
1	∞	2
2	∞	2.742
3	12.15	2.981
4	∞	2.982
5	46.875	2.997
6	27.771	2.996
7	30.012	2.999
8	∞	2.998

Conjecture:

$$\dim(E_1(I_m)) = \begin{cases} \dfrac{\lvert V(I_m)\rvert}{3} + 2, \text{ when } m \text{ is odd} \\ \dfrac{\lvert V(I_m)\rvert}{3} + 6, \text{ when } m \text{ is even} \end{cases}$$



Result

$$I_m = \mathbb{Z}_m^4\text{-cover of } \square$$

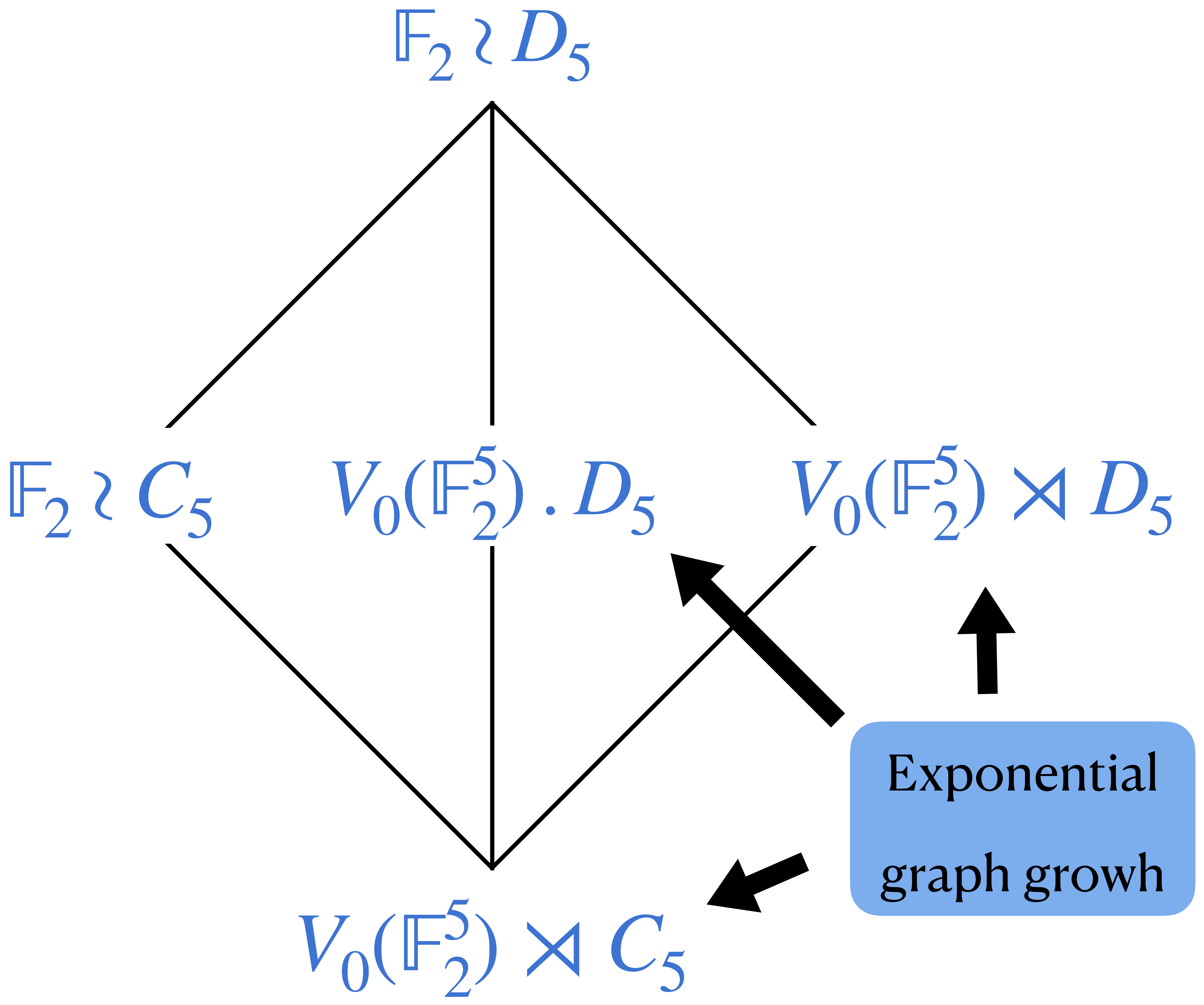
(I_m, H_m) is locally-C_5 and (I_m, G_m) is locally-D_5

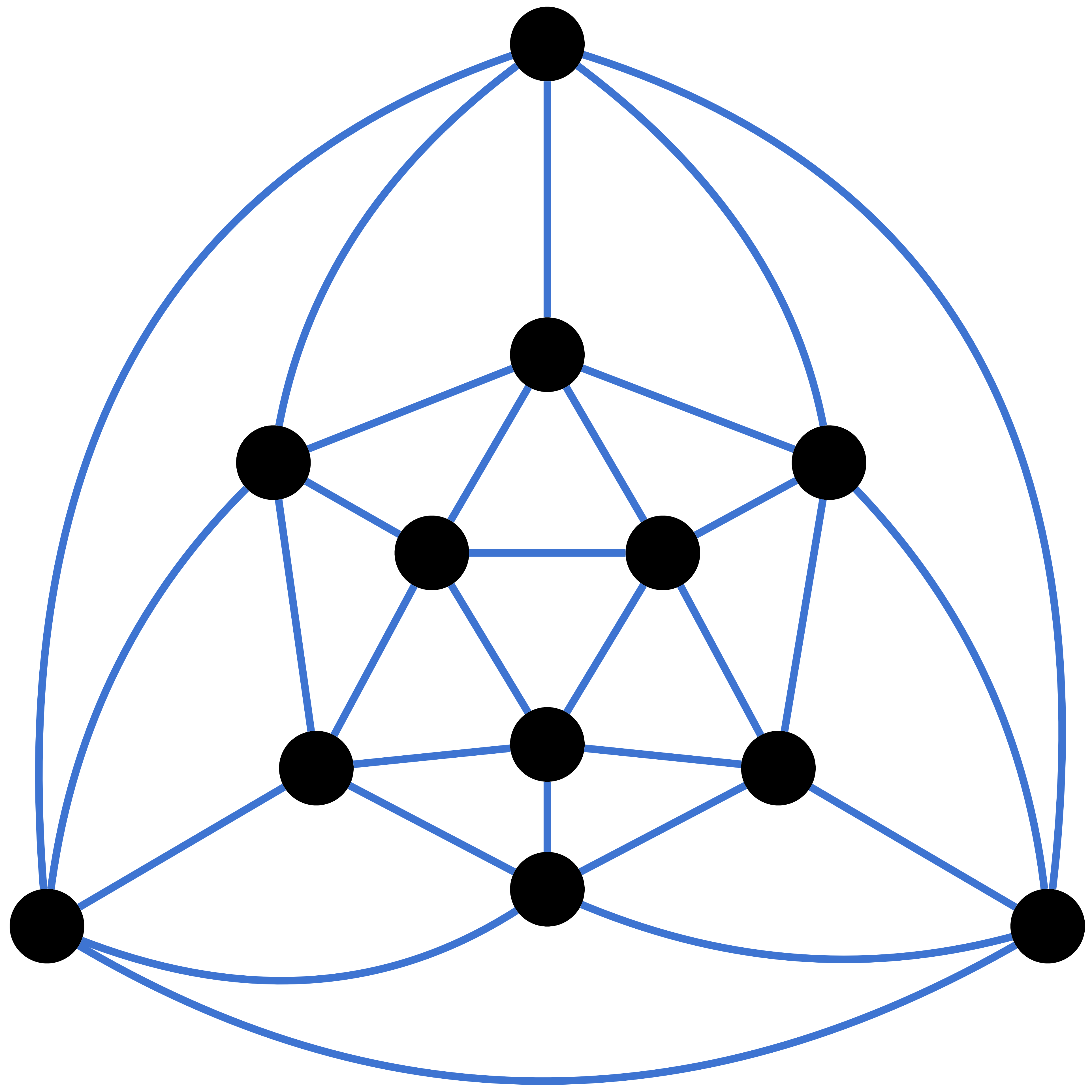
H_m is normal in G_m

$$\dim_{\mathbb{F}_2}(E_1(I_m)) \geq \frac{|V(I_m)|}{12}$$

f_m is the lift of $f \in \mathbb{F}_2^{V(\square)}$ such that $f^x - f \in E_1(\square), x \in H$

Graph growth of some groups of degree 10





Work in progress

Recent developments: Twisting Vectors

X = graph, $G \leq \mathrm{Aut}(X)$ is transitive on vertices

$\mathbb{F}$ = field, $\lambda \in \mathbb{F}$

$f \in \mathbb{F}^{V(X)}$ such that $f^x - f \in E_\lambda(X), \forall x \in G$

Proposition

A twisting vector exists if and only if the following equation has a solution

$$(A(X) - \lambda I)x = \vec{1}$$

Recent developments: Twisting Vectors (over $\mathbb{F}_2$)

A twisting vector exists if and only if the following equation has a solution

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	even valency	odd valency
$\lambda = 0$		$\vec{1}$
$\lambda = 1$	$\vec{1}$	

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$\lambda = 0$	May not exist! (recall T_{30})	$\vec{1}$
$\lambda = 1$	$\vec{1}$	Always exists!

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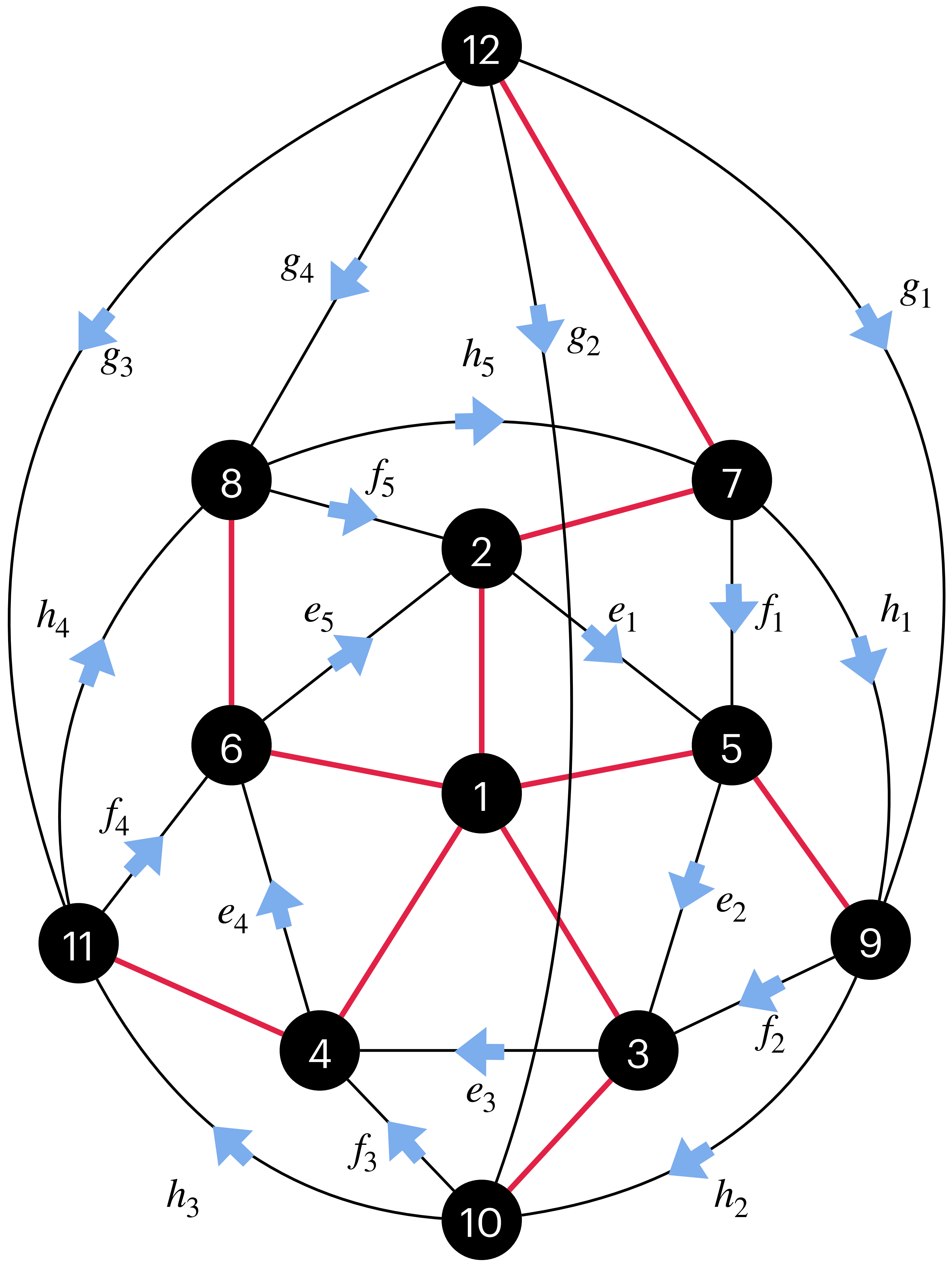
FACT

If $A \in M_n(\mathbb{F}_2)$ is symmetric, then im(A) contains the diagonal of A.

New description of the exponential family



Sabidussi double-coset graphs

$$G_n = V_0(\mathbb{Z}_n^5) \rtimes A_5$$

$$H_n = 1 \times \langle (1,2,3,4,5) \rangle \cong C_5$$

$$a_n = ([1,-1,1,-1,0], (1,2)(3,4)) \in G_n$$

Infinite family of graphs I_n with

$V(I_n)$ = right cosets of H_n in G_n

$$E(I_n) = H_n x \sim H_n y \iff xy^{-1} \in H_n a_n H_n$$

Graph growth of low degree groups

Determining the graph growth of the last transitive permutation group of degree 8

The eigenspace technique

Given a permutation group L, how to construct locally-L graphs with large eigenspaces over finite fields?

$$\dim_{\mathbb{F}}(E_\lambda(X_m)) \geq c\,|V(X_m)|$$

Future work

Polynomial bounds on the graph growth of permutation groups

Do there exist permutation groups of "intermediate" graph growth?

Questions?

Thank you for your attention!