Aldous' spectral gap conjecture and generalizations

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Outline

1. Three versions of Aldous' spectral gap conjecture
2. Representation theory of symmetric groups
3. Generalizations

Three versions of Aldous’ spectral gap conjecture

Notations

- Let $\Gamma = (V(\Gamma), E(\Gamma), w)$ be a connected and weighted graph with

$$V(\Gamma) = \{v_1, v_2, \ldots, v_n\},$$
$$E(\Gamma) \subseteq \big\{\{v_i, v_j\} \mid v_i \neq v_j \in V(\Gamma)\big\},$$
and $w : E(\Gamma) \to \mathbb{R}^+,\ \{v_i, v_j\} \mapsto w(\{v_i, v_j\})$.

- Consider w as a map on $[n] \times [n]$ by letting

$$w_{ij} = w_{ji} = w(\{v_i, v_j\}) > 0, \quad \forall\ \{v_i, v_j\} \in E(\Gamma),$$
and $w = 0$ everywhere else,

where $[n] = \{1, 2, \ldots, n\}$.

- Any unweighted graph can be seen as a weighted graph with every edge having weight 1.

Aldous' spectral gap conjecture

Aldous' spectral gap conjecture (1992, [1])

For any unweighted and connected graph Γ, the random walk and the interchange process on Γ have the same relaxation time.

- This conjecture indicates that these two Markov chains converge to their respective stationary distributions at the same asymptotic rate.
- In 2010, this conjecture was confirmed in its general form for any weighted and connected graph Γ (see [2]).

1. D. Aldous, https://www.stat.berkeley.edu/∼aldous/Research/OP/sgap.html.
2. P. Caputo, T. M. Liggett, and T. Richthammer. Proof of Aldous' spectral gap conjecture. J. Amer. Math. Soc., 23(3):831-851, 2010.

Notations

- The adjacency matrix of Γ is $A(\Gamma) = (a_{ij})_{n\times n}$ with $a_{ij} := w_{ij}$.
- The eigenvalues of Γ are defined to be the those of $A(\Gamma)$, denoted by

$$\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma).$$

- Γ is said to be regular with degree k if

$$\sum_{j\in[n]} w_{ij} = k, \quad \forall\, i \in [n].$$

- When Γ is connected and regular with degree k, we have

$$\lambda_1(\Gamma) = k \text{ and } \lambda_2(\Gamma) < k,$$

and $\lambda_1(\Gamma) - \lambda_2(\Gamma)$ is called the spectral gap of Γ.

Cayley graphs

- Let G be a finite group with the identity element e.

  A symmetric weighted subset S of G satisfies

  $$e \notin S, \quad S = S^{-1} := \{s^{-1} \mid s \in S\},$$

  $$\text{and } w_s = w_{s^{-1}} > 0, \quad \forall\, s \in S.$$

  The weight of S is defined to be $|S| := \sum_{s \in S} w_s$.

- The Cayley graph $\mathrm{Cay}(G, S)$ is the weighted graph with

  $$V = G, \quad E = \big\{\{g, gs\} \mid g \in G, s \in S\big\},$$

  and w_s as the weight of the edge $\{g, gs\}$.

  We call S the connection set of $\mathrm{Cay}(G, S)$ and

  $$\mathrm{Cay}(G, S) \text{ is connected iff } G = \langle S \rangle.$$

- $\mathrm{Cay}(G, S)$ is regular with degree $|S|$.

Aldous' spectral gap conjecture

- Let S_n be the symmetric group on $[n]$.
- $(S_n)_1 = \{\sigma \in S_n \mid \sigma(1) = 1\}$ is the stabilizer of $1 \in [n]$ in S_n.

Aldous' spectral gap conjecture V.2

Let T be any weighted generating subset of S_n consisting of transpositions. Then the weighted graph $\Gamma = \mathrm{Cay}(S_n, T)$ has the same spectral gap as $\Gamma' = \mathrm{Sch}(S_n, (S_n)_1, T)$.

- $\Gamma' = \mathrm{Sch}(S_n, (S_n)_1, T)$ is the Schreier coset graph with

  $$V = \{H_i \mid H_i \text{ is a right coset of } (S_n)_1 \text{ in } S_n, 1 \leq i \leq n\},$$

  and $\sum_{\tau \in T,\ H_j = H_i\tau} w_\tau$ as the weight of $\{H_i, H_j\}$.

- Γ and Γ' are both regular with degree $|T| \Rightarrow \lambda_1(\Gamma) = \lambda_1(\Gamma') = |T|$.

Representation theory of symmetric groups

Representation theory of finite groups

Definition [3]

A ***matrix representation*** of a finite group G is a group homomorphism

$$\phi : G \to \mathrm{GL}_d,$$

where GL_d is the complex general linear group of degree d. The parameter d is called the ***degree*** of ϕ.

Definition [3]

Let V be a vector space over $\mathbb{C}$ and of finite dimension $\dim(V)$ and G be a finite group. Then V is a ***G-module*** if there is a group homomorphism

$$\rho : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is the general linear group of V.

- We are using $g\mathbf{v}$ as a shorthand for the application of the transformation $\rho(g)$ to the vector $\mathbf{v}$.

3. B. Sagan. The symmetric group: representations, combinatorial algorithms, and symmetric functions. Vol. 203. Springer Science & Business Media, 2001.

G-equivalence

Definition [3]

Let W and V be G-modules. Then a ***G-homomorphism*** (or simply a homomorphism) is a linear transformation $\theta : W \to V$ such that

$$\theta(g\mathbf{w}) = g\theta(\mathbf{w})$$

for all $g \in G$ and $\mathbf{w} \in W$. If $\theta : W \to V$ is also bijective, then we say θ is a ***G-isomorphism***. In this case we say that W and V are ***G-isomorphic***, or ***G-equivalent***, written $W \cong V$. Otherwise we say that W and V are ***G-inequivalent***.

- Matrix representations X and Y of a group G are equivalent if and only if there exists a fixed matrix T such that

$$Y(g) = TX(g)T^{-1} \quad \text{for all } g \in G.$$

Representation theory of finite groups

Example [3]

The ***trivial representation*** of G sends every $g \in G$ to the matrix (1).

Example [3]

A finite group G acts on itself by right multiplication:

if $g, h \in G$, the action of g on h equals hg^{-1}.

The ***(right) regular representation*** $\mathbf{R_G}$ of G is of degree $|G|$ and for every $g \in G$,

$$R_G(g) = (r_{s,h})_{|G|\times|G|} \text{ with } r_{s,h} = \begin{cases} 1, & \text{if } hg^{-1} = s; \\ 0, & \text{otherwise.} \end{cases}$$

Adjacency matrices of Cayley graphs

- For any weighted subset $S \subset G$,

$$A\big(\mathrm{Cay}(G,S)\big) = \sum_{g \in S} w_g R_G(g).$$

The (s,h)-entry of the left is

$$\sum_{\substack{g \in S \\ h = sg}} w_g.$$

The (s,h)-entry of the right is

$$\sum_{g \in S} w_g \big(R_G(g)\big)_{s,h} = \sum_{\substack{g \in S \\ hg^{-1} = s}} w_g$$

- We use R_n to indicate the (right) regular representation of S_n. Then for any weighted subset S of S_n, we have

$$A(\mathrm{Cay}(S_n,S)) = \sum_{\sigma \in S} w_\sigma R_n(\sigma).$$

Example [3]

The map $\phi(\sigma) = \mathrm{sgn}(\sigma),\ \forall\ \sigma \in S_n$ is a nontrivial representation of S_n with degree 1, called the ***sign representation of $\mathbf{S_n}$***.

Example [3]

The ***defining representation $\mathbf{D_n}$*** of S_n is of degree n and for all $\sigma \in S_n$,

$$D_n(\sigma) = (d_{i,j})_{n\times n} \text{ with } d_{i,j} = \begin{cases} 1, & \text{if } \sigma(j) = i; \\ 0, & \text{otherwise.} \end{cases}$$

Adjacency matrix of $\mathrm{Sch}(\mathrm{S_n}, (\mathrm{S_n})_1, \mathrm{S})$

- For any weighted subset $S \subset S_n$,

$$A\big(\mathrm{Sch}(S_n, (S_n)_1, S)\big) = \sum_{\sigma \in S} w_\sigma D_n(\sigma).$$

The (i, j)-entry of the left is

$$\sum_{\substack{\sigma \in S \\ H_j = H_i \sigma}} w_\sigma.$$

The (i, j)-entry of the right is

$$\sum_{\sigma \in S} w_\sigma \big(D_n(\sigma)\big)_{i,j} = \sum_{\substack{\sigma \in S \\ \sigma(j) = i}} w_\sigma.$$

Submodules

Definition [3]

Let V be a G-module. A ***submodule*** of V is a subspace W that is closed under the action of G, i.e.,

$$\mathbf{w} \in W \Rightarrow g\mathbf{w} \in W \text{ for all } g \in G.$$

Equivalently, W is a subset of V that is a G-module in its own right. We write $W \leq V$ if W is a submodule of V.

Example [3]

Any G-module, V, has the submodules $W = V$ as well as $W = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector. These two submodules are called ***trivial***. All other submodules are called ***nontrivial***.

Irreducible Representations

Definition [3]

A nonzero G-module V is ***reducible*** if it contains a non-trivial submodule W. Otherwise, V is said to be ***irreducible***. Equivalently, V is ***reducible*** if it has a basis $\mathcal{B}$ in which every $g \in G$ is assigned a block matrix of the form $X(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}$, where the $A(g)$ are square matrices, all of the same size, and 0 is a nonempty matrix of zeros.

Definition [3]

Let V be a vector space with subspaces U and W. Then V is the ***(internal) direct sum*** of U and W, written $V = U \oplus W$, if every $\mathbf{v} \in V$ can be written uniquely as a sum $\mathbf{v} = \mathbf{u} + \mathbf{w}$, $\mathbf{u} \in U$, $\mathbf{w} \in W$. If X is a matrix, then X is the ***direct sum*** of matrices A and B, written $X = A \oplus B$, if X has the block diagonal form $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Maschke's Theorem

Maschke's Theorem [3]

Let G be a finite group and let V be a nonzero G-module. Then

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)},$$

where each $W^{(i)}$ is an irreducible G-submodule of V.

Maschke's Theorem [3]

Let G be a finite group and let X be a matrix representation of G of degree $d > 0$. Then there is a fixed matrix T such that every matrix $X(g), g \in G$, has the form

$$TX(g)T^{-1} = \begin{pmatrix} X^{(1)}(g) & 0 & \cdots & 0 \\ 0 & X^{(2)}(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X^{(k)}(g) \end{pmatrix},$$

where each $X^{(i)}$ is an irreducible matrix representation of G.

Complete Reducibility

Definition [3]

A representation is ***completely reducible*** if it can be written as a direct sum of irreducible representations.

So Maschke's theorem could be restated:

Maschke's Theorem [3]

Every (complex) representation of a finite group having positive dimension is completely reducible.

Characters

Definition [3]

Let $X(g),\ g \in G$, be a matrix representation. Then the ***character*** of X is

$$\chi(g) = \mathrm{tr}\ X(g),$$

where tr denotes the trace of a matrix. In other words, χ is the map

$$G \quad \xrightarrow{\mathrm{tr}\ X} \quad \mathbb{C}.$$

If V is a G-module, then its character is the character of a matrix representation X corresponding to V.

If X and Y both correspond to V, then $Y = TXT^{-1}$ for some fixed T. Thus, for all $g \in G$,

$$\mathrm{tr}\ Y(g) = \mathrm{tr}\ TX(g)T^{-1} = \mathrm{tr}\ X(g).$$

Characters

If X has character χ, we say that χ is irreducible whenever X is.

Example [3]

We consider the defining representation of S_n with its character χ^{def}. It is not hard to see that if $\sigma \in S_n$, then

$$\chi^{\text{def}}(\sigma) = \text{the number of fixedpoints of } \sigma.$$

Example [3]

Consider the regular representation of G and denote its character by χ^{reg}. Then we have

$$\chi^{\text{reg}}(g) = \begin{cases} |G|, & \text{if } g = e; \\ 0, & \text{otherwise.} \end{cases}$$

Characters

Proposition [3]

Let X be a matrix representation of G of degree d with character χ.

1. $\chi(e) = d$.
2. If K is a conjugacy class of G, then

$$g, h \in K \implies \chi(g) = \chi(h).$$

3. For any $g \in G$, $\chi(g^{-1}) = \overline{\chi(g)}$.

Definition [3]

Let χ and ϕ be any two functions from a group G to the complex numbers $\mathbb{C}$. The ***inner product*** of χ and ϕ is

$$\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\phi(g)}.$$

Inner Products of Characters

Proposition [3]

Let χ and ϕ be characters; then

$$\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(g^{-1}).$$

Character Relations of the First Kind [3]

If χ and ϕ are irreducible characters of G, then

$$\langle \chi, \phi \rangle = \delta_{\chi,\phi}.$$

Character Relations of the First Kind

Corollary [3]

Let X be a matrix representation of G with character χ. Suppose

$$X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)},$$

where the $X^{(i)}$ are pairwise inequivalent irreducibles with characters $\chi^{(i)}$ and $m_i X^{(i)} = \underbrace{X^{(i)} \oplus X^{(i)} \oplus \cdots \oplus X^{(i)}}_{m_i}$.

1. $\chi = m_1 \chi^{(1)} + m_2 \chi^{(2)} + \cdots + m_k \chi^{(k)}$.
2. $\langle \chi, \chi^{(j)} \rangle = m_j$ for all j.
3. $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \cdots + m_k^2$.
4. X is irreducible if and only if $\langle \chi, \chi \rangle = 1$.
5. Let Y be another matrix representation of G with character ϕ. Then

   $$X \cong Y \text{ if and only if } \chi(g) = \phi(g)$$

   for all $g \in G$.

Decomposition of the regular representation R_G

Proposition [3]

Let G be a finite group and suppose $R_G = \bigoplus_i m_i X^{(i)}$, where the $X^{(i)}$ form a complete list of pairwise inequivalent irreducible matrix representations of G. Then

1. $m_i = \deg X^{(i)}$,
2. $\sum_i (\deg X^{(i)})^2 = |G|$, and
3. The number of $X^{(i)}$ equals the number of conjugacy classes of G.

Eigenvalues of Cayley Graphs

Theorem [4]

Let G be a finite group and S a (symmetric) weighted subset of G. $\widehat{G} = \{X^{(1)}, X^{(2)}, \cdots, X^{(k)}\}$ is a complete set of inequivalent (complex) irreducible matrix representations of G. Then the adjacency matrix of $\mathrm{Cay}(G,S) = \sum_{s \in S} w_s R_G(s)$ is similar to

$$d_1 X^{(1)}(S) \oplus d_2 X^{(2)}(S) \oplus \cdots \oplus d_k X^{(k)}(S),$$

where d_i is the degree of $X^{(i)}$, $X^{(i)}(S) = \sum_{s \in S} w_s X^{(i)}(s)$, and

$$d_i X^{(i)}(S) = \underbrace{X^{(i)}(S) \oplus X^{(i)}(S) \oplus \cdots \oplus X^{(i)}(S)}_{d_i}.$$

4. M. Krebs and A. Shaheen. Expander families and Cayley graphs: a beginner's guide. Oxford Univ. Press, 2011.

Representation theory of $\mathbf{S_n}$

Definition [3]

A sequence of positive integers $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$, satisfying $\gamma_1 \geq \gamma_2 \geq \dots \gamma_m > 0$ and $n = \sum_{i=1}^{m} \gamma_i$, is called a ***partition*** of n, denoted by $\gamma \vdash n$.

- For each $\gamma \vdash n$, there is a corresponding irreducible representation ρ_γ of S_n, known as the **Specht module**, with degree d_γ.
- $\widehat{S_n} := \{\rho_\gamma \mid \gamma \vdash n\}$ is a complete set of inequivalent irreducible matrix representations of S_n.
- $\rho_{(n)}$: trivial representation of S_n with $d_{(n)} = 1$

  $\rho_{(1^n)}$: sign representation of S_n with $d_{(1^n)} = 1$

  $\rho_{(n-1,1)}$: **standard representation** of S_n with $d_{(n-1,1)} = n - 1$

Representation theory of $\mathrm{S_n}$

Decomposition of $\mathrm{R_n}$ [3]

For the regular representation R_n of S_n, there is a fixed matrix M such that

$$MR_n(\sigma)M^{-1} = \bigoplus_{\gamma \vdash n} d_\gamma \rho_\gamma(\sigma), \quad \forall\, \sigma \in S_n.$$

Here $d_\gamma \rho_\gamma(\sigma) = \underbrace{\rho_\gamma(\sigma) \oplus \rho_\gamma(\sigma) \oplus \cdots \oplus \rho_\gamma(\sigma)}_{d_\gamma}$.

Decomposition of $\mathrm{D_n}$ [3]

For the defining representation D_n of S_n, there is a fixed matrix M such that

$$MD_n(\sigma)M^{-1} = \rho_{(n)}(\sigma) \bigoplus \rho_{(n-1,1)}(\sigma), \quad \forall \sigma \in S_n.$$

Application of representation theory

- Let S be any weighted subset of S_n and

$$\rho_\gamma(S) := \sum_{\sigma \in S} w_\sigma \rho_\gamma(\sigma), \quad \forall\, \gamma \vdash n \tag{1}$$

- $A\big(\mathrm{Cay}(S_n, S)\big) = \sum_{\sigma \in S} w_\sigma R_n(\sigma)$ is similar to

$$\bigoplus_{\gamma \vdash n} d_\gamma \rho_\gamma(S). \tag{2}$$

$A\big(\mathrm{Sch}(S_n, (S_n)_1, S)\big) = \sum_{\sigma \in S} w_\sigma D_n(\sigma)$ is similar to

$$\rho_{(n)}(S) \bigoplus \rho_{(n-1,1)}(S). \tag{3}$$

- If $\gamma = (n)$, we have $d_\gamma = 1$ and $d_\gamma \rho_\gamma(S) = \rho_\gamma(S) = |S|$.

When S is symmetric and generates S_n,

$$\lambda_2(\mathrm{Cay}(S_n, S)) = \max_{\substack{\gamma \vdash n \\ \gamma \neq (n)}} \lambda_1(\rho_\gamma(S)) < |S|, \tag{4}$$

$$\lambda_2(\mathrm{Sch}(S_n, (S_n)_1, S)) = \lambda_1(\rho_{(n-1,1)}(S)) < |S|. \tag{5}$$

Aldous property

Aldous' Spectral Gap Conjecture V.3

Let T be any weighted generating subset of S_n consisting of transpositions. The second largest eigenvalue of $\mathrm{Cay}(S_n, T)$ is obtained by the standard representation $\rho_{(n-1,1)}$, that is,

$$\lambda_2(\mathrm{Cay}(S_n, T)) = \lambda_1(\rho_{(n-1,1)}(T)).$$

In general, the ***strictly second largest eigenvalue*** of a regular graph is the largest eigenvalue strictly smaller than the degree of the graph.

Definition [7]

For any symmetric weighted subset $S \subset S_n$, we say that $\mathrm{Cay}(S_n, S)$ has the ***Aldous property*** if its strictly second largest eigenvalue is attained by the standard representation of S_n, that is,

$$\lambda_{t+1}(\mathrm{Cay}(S_n, S)) = \lambda_1(\rho_{(n-1,1)}(S)),$$

where $t := [S_n : \langle S \rangle]$ is the index of $\langle S \rangle$ in S_n.

The normal case of Aldous' Spectral Gap Conjecture

Definition

For any symmetric weighted subset S of a group G, Cay(G, S) is said to be ***normal*** if the weights of S are constant on any conjugacy class of G, that is, $w_s = w_{g^{-1}sg}$, for any $g \in G$ and $s \in S$.

- $\Gamma = \mathrm{Cay}(S_n, T)$ is normal if and only if

  T consists of all transpositions in S_n,

  and $w_\tau = c$ for every $\tau \in T$,

  where c is a fixed positive integer. In this case, we can regard $\Gamma = \mathrm{Cay}(S_n, T)$ as a unweighted graph.

Eigenvalues of normal Cayley graphs

Proposition [5, 6]

Let $\{\chi_1, \chi_2, \ldots, \chi_k\}$ be a complete set of inequivalent irreducible characters of G. Then the eigenvalues of any weighted normal Cayley graph $\mathrm{Cay}(G, S)$ on G are given by

$$\lambda_j = \frac{1}{\chi_j(e)} \sum_{s \in S} w_s \chi_j(s) = \sum_{s \in S} w_s \tilde{\chi}_j(s), \quad j = 1, 2, \ldots, k.$$

Moreover, the multiplicity of λ_j is equal to $\sum_{1 \le i \le k,\ \lambda_i = \lambda_j} \chi_i(e)^2$.

5. P. Diaconis and M. Shahshahani. Generating a random permutation with random transpositions. Z. Wahrscheinlichkeitstheor. Verw. Geb., 57(2):159-179, 1981.
6. P. H. Zieschang. Cayley graphs of finite groups. J. Algebra, 118(2):447-454, 1988.

Generalizations

Normal Cayley graphs having the Aldous property

Definition

For any $\sigma \in S_n$, the ***support*** of σ is defined to be

$$\operatorname{supp}(\sigma) := \{i \in [n] \mid \sigma(i) \neq i\}.$$

For $\emptyset \neq I \subseteq \{2, 3, \ldots, n-1, n\}$ and $2 \leq k \leq n$, set

$$T(n, I) = \{\sigma \in S_n \mid |\operatorname{supp}(\sigma)| \in I\}$$

and

$$T(n, k) = \{\sigma \in S_n \mid 2 \leq |\operatorname{supp}(\sigma)| \leq k\}.$$

Normal Cayley graphs having the Aldous property

Theorem 1 [7]

There exists a positive integer N such that for every $n \geq N$ and any conjugacy class S of S_n, the normal Cayley graph $\mathrm{Cay}(S_n, S)$ has the Aldous property if and only if $2 \leq |\mathrm{supp}(\sigma)| \leq n-2$ for some (and hence all) $\sigma \in S$.

Theorem 2 [7]

There exists a positive integer N such that for every $n \geq N$ and any $\emptyset \neq I \subset \{2, 3, \ldots, n-1, n\}$ with $|I \cap \{n-1, n\}| \neq 1$, the normal Cayley graph $\mathrm{Cay}(S_n, T(n, I))$ has the Aldous property if and only if $I \cap \{n-1, n\} = \emptyset$.

7. Y. Li, B. Xia and S. Zhou. Aldous' spectral gap property for normal Cayley graphs on symmetric groups. Submitted. 2022.

Normal Cayley graphs having the Aldous property

Theorem 3 [7]

There exists a positive integer N such that for every $n \geq N$ and any $2 \leq k \leq n$, the connected normal Cayley graph $\mathrm{Cay}(S_n, T(n,k))$ has the Aldous property.

- Theorem 3 generalizes the normal case of Aldous' conjecture.
- $S(n,k) := \{\sigma \in S_n \mid \sigma \text{ fixes } k \text{ elements}\}$ with $0 \leq k \leq n-2$
- The k-point fixing graph is defined to be $\mathcal{F}(n,k) = \mathrm{Cay}(S_n, S(n,k))$.

Corollary 4 [7]

There exists a positive integer N such that for every $n \geq N$, the k-point-fixing graph $\mathcal{F}(n,k)$ has the Aldous property if and only if $2 \leq k \leq n-2$.

7. Y. Li, B. Xia and S. Zhou. Aldous' spectral gap property for normal Cayley graphs on symmetric groups. Submitted. 2022.

Normal Cayley graphs having the Aldous property

Problem 5 [7]

Give a necessary and sufficient condition for $\mathrm{Cay}(S_n, T(n, I))$ with $\{n-1\} \subset I \subset \{2, 3, \ldots, n-2, n-1\}$ to have the Aldous property for sufficiently large n.

Problem 6 [7]

Give a necessary and sufficient condition for $\mathrm{Cay}(S_n, T(n, I))$ with $\{n\} \subset I \subseteq \{2, 3, \ldots, n-2, n\}$ to have the Aldous property for sufficiently large n.

7. Y. Li, B. Xia and S. Zhou. Aldous' spectral gap property for normal Cayley graphs on symmetric groups. Submitted. 2022.

Normal Cayley graphs having the Aldous property

Proposition 2.3 in [9]

There exists some positive integer N such that for every $n \geq N$ and any $\sigma \in S_n$ with $|\mathrm{fix}(\sigma)| \geq 2$, we have

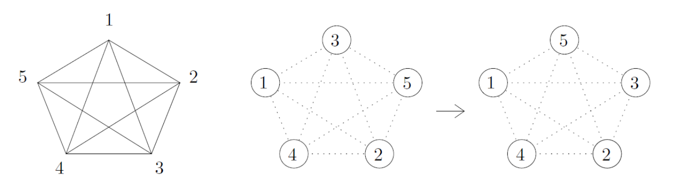
$$\max_{\substack{\gamma \vdash n \\ \gamma \neq (n) \text{ or } (1^n)}} \tilde{\chi}_\gamma(\sigma) = \tilde{\chi}_{(n-1,1)}(\sigma). \tag{6}$$

[9.] O. Parzanchevski and D. Puder. Aldous's spectral gap conjecture for normal sets. Trans. Amer. Math. Soc., 373(10):7067–7086, 2020.

Thank you for your attention.

Interchange process

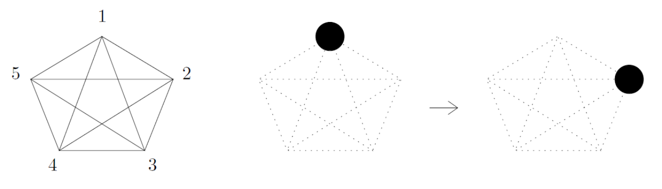
- A state is identical to an element of the symmetric group S_n.
- The transition from a sate $\sigma \in S_n$ to a state σ^{ij}, occurring with rate w_{ij}, interchanges the particles at vertices i and j.
- The state σ^{ij} is $\sigma\tau_{ij}$, where τ_{ij} denotes the transposition $(i,j) \in S_n$.



A transition from the state $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ to $\sigma^{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$

Random walk

- The random walk on Γ is the continuous-time Markov chain in which a single particle jumps from the vertex $i \in V(\Gamma)$ to j at rate w_{ij}.
- The state space of this random walk is $V(\Gamma) = \{1, 2, \ldots, n\}$.



A transition from the state 1 to the state 2